Basics of Large Eddy Simulation

by

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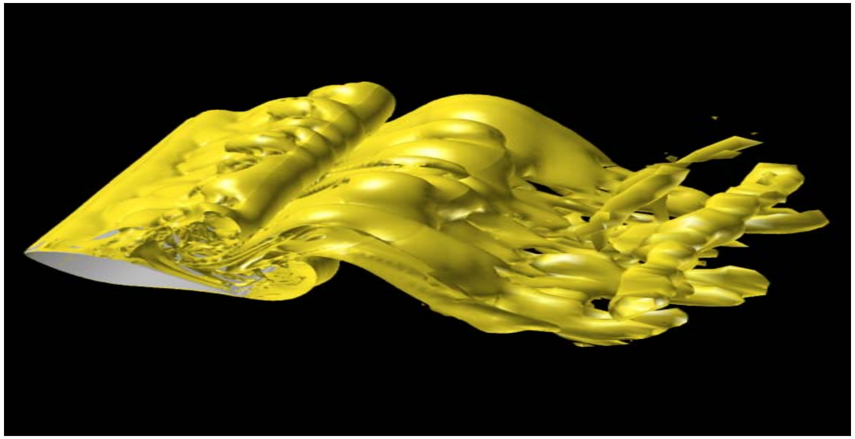
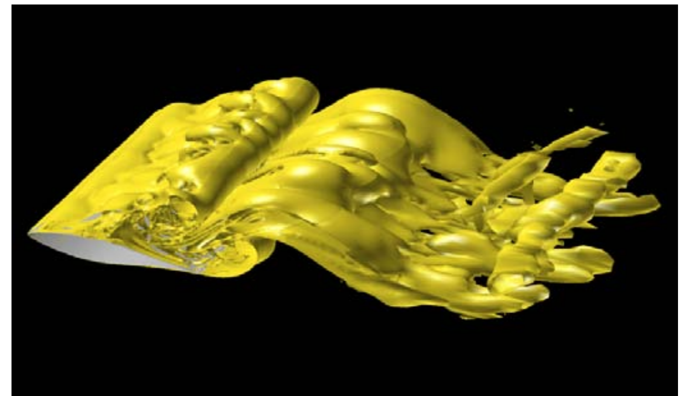


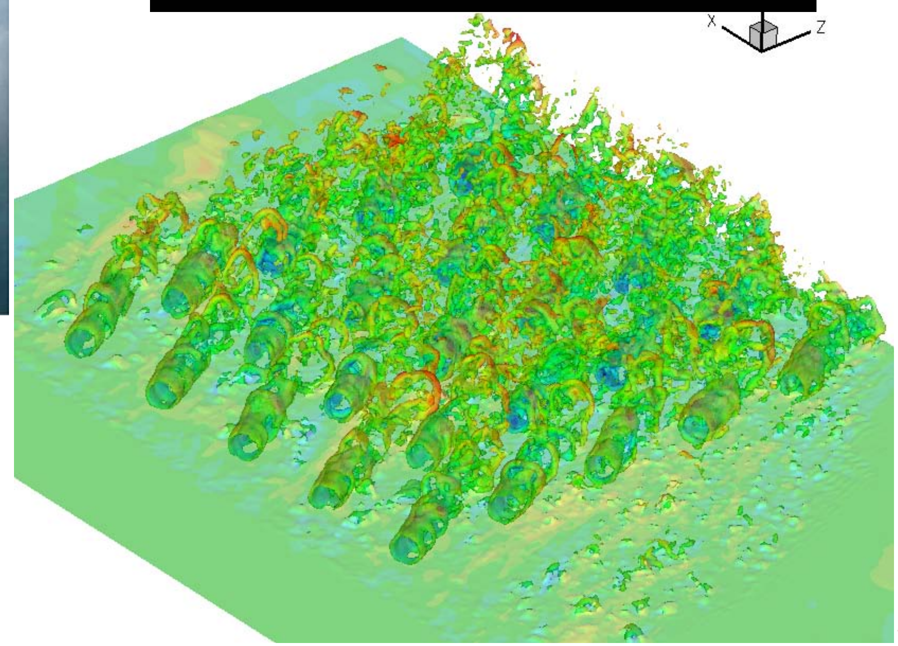
LES and turbulence

Characteristics of turbulence:

- **Random fluctuations**
- **It contains eddies of many sizes (from the shear layer thickness δ to the Kolmogorov length scale)**
- **Self sustaining**
- **Generated by velocity gradients**
- **Dissipative**
- **Mixing**

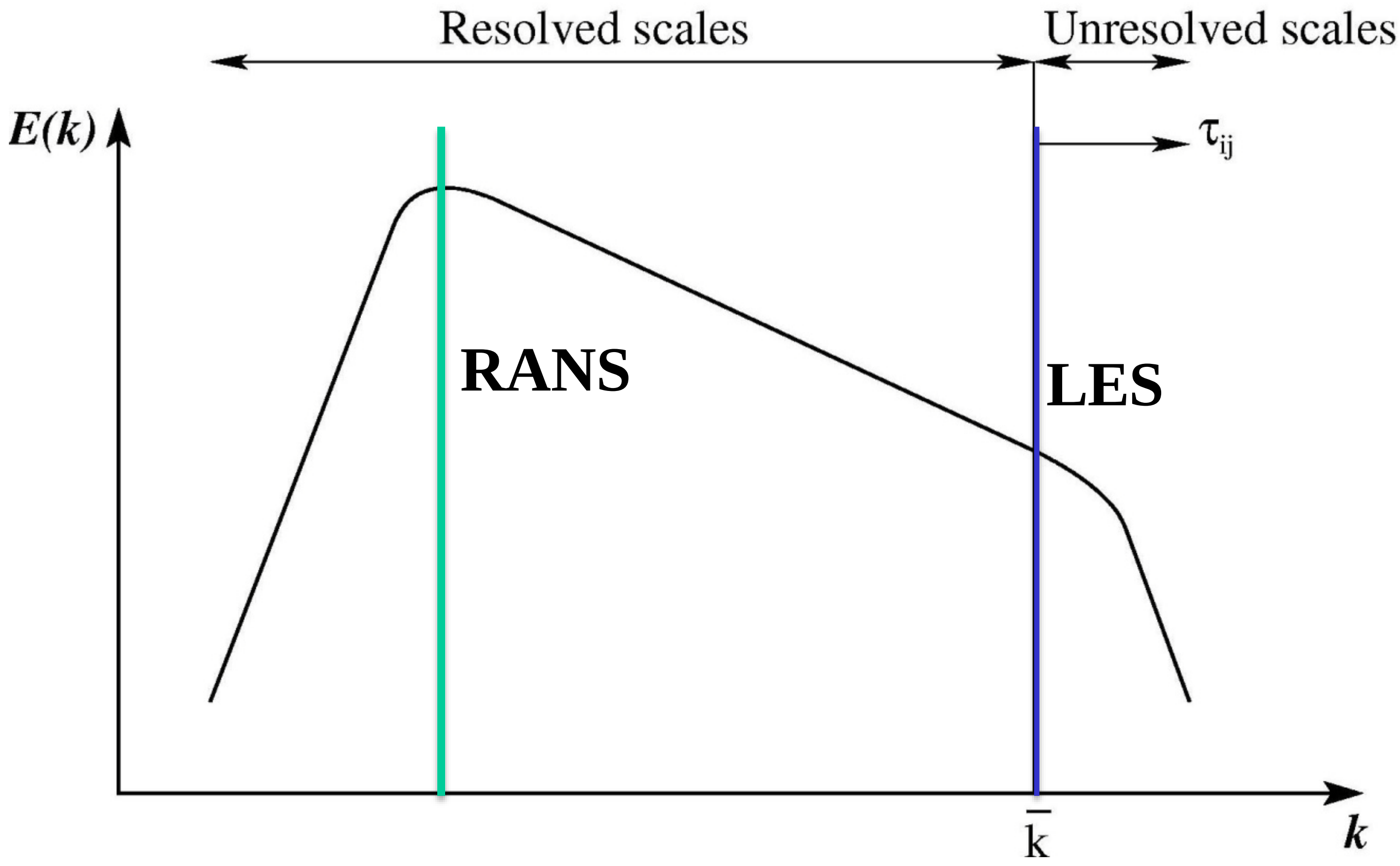
Turbulent wakes in wind energy





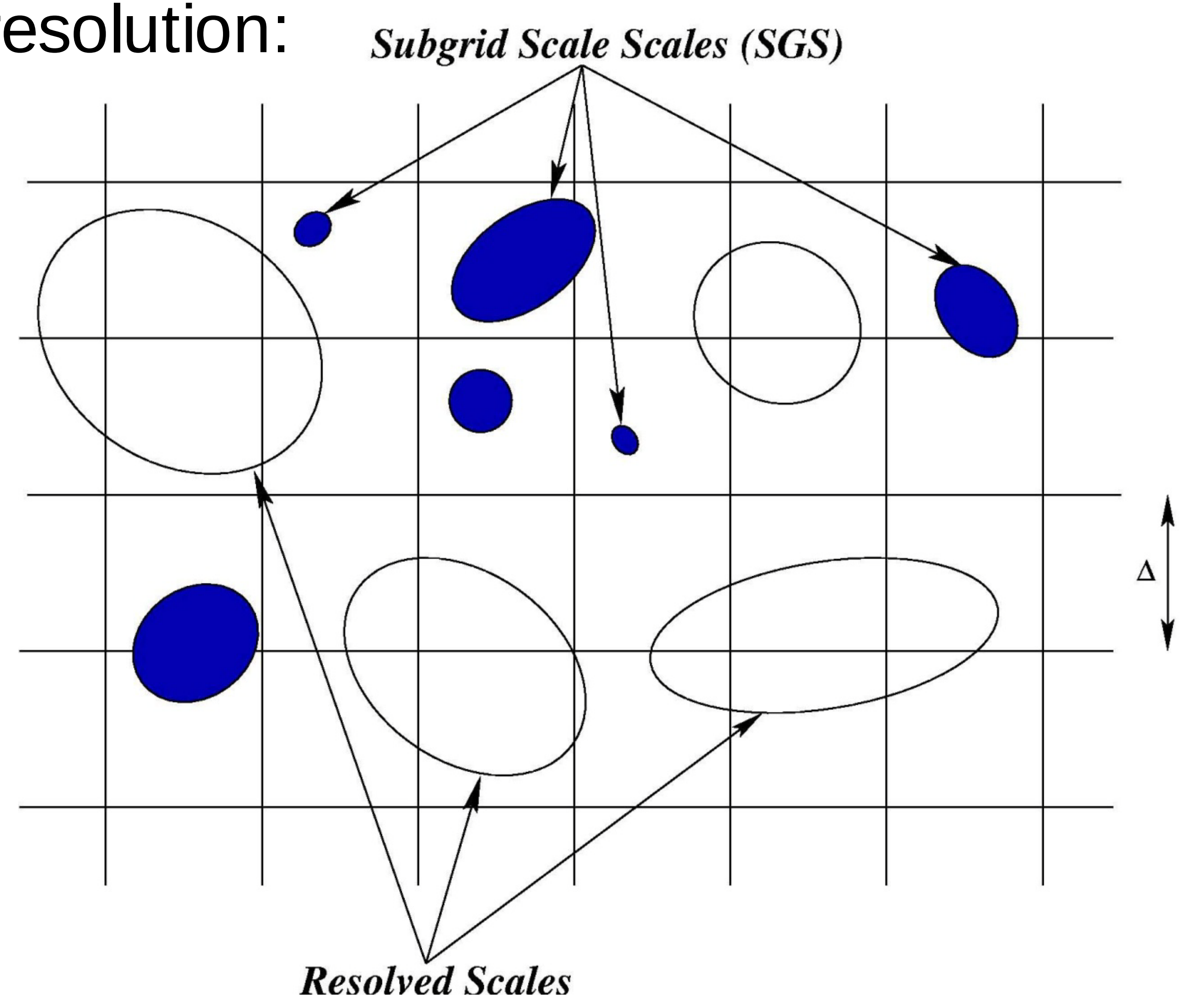
LES and turbulence

Turbulence spectrum:



LES and turbulence

Flow resolution:



Scale requirements in wind energy

Turbulent scales:

	Length scale (m)	Velocity scale (m/s)	Time scale (s)
Airfoil boundary layer	10^{-3}	10^{2}	10^{-5}
Airfoil	1	10^{2}	10^{-2}
Rotor	10^{2}	10	10
Cluster	10^{3}	10	10^{2}
Wind farm	10^{4}	10	10^{3}

LES in Wind Energy

Why not use DNS:

Smallest turbulent length scale: ℓ
Largest geometrical length scale: L

Estimate based on scales:
$L/\ell \approx Re^{(3/4)}$

Reynolds Number: $Re = L \cdot U/\nu$,
where
L: Length of object
U: Typical velocity
ν: Kinematic viscosity

Number of mesh points: $N \approx (L/\ell)^{3} = Re^{(9/4)}$

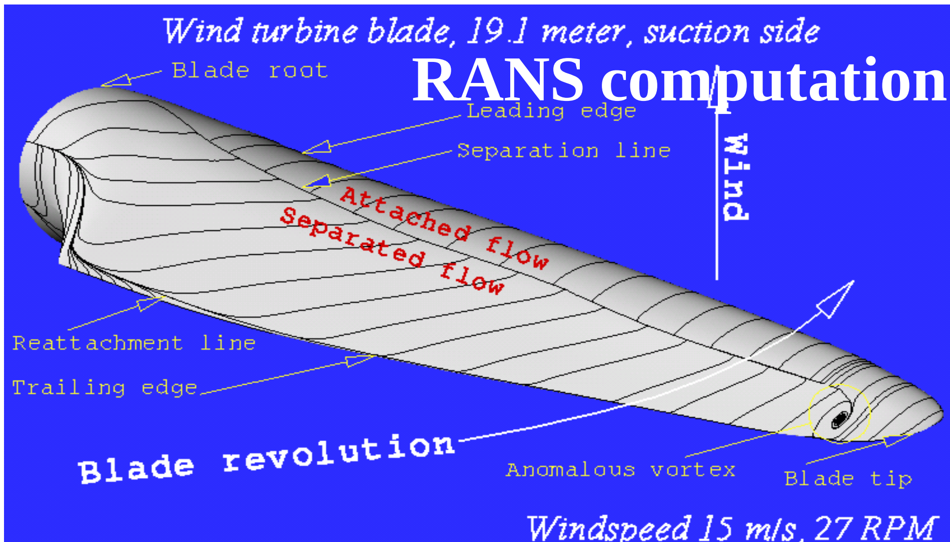
Typically Re = O(10**6) - O(10**7), thus N = O(10**15)

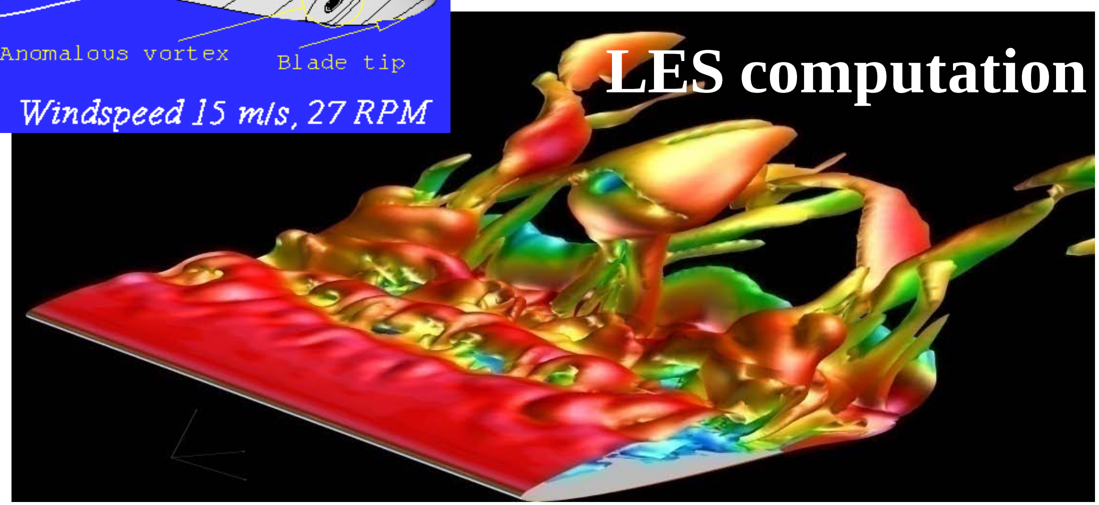
Computing performance is generally 10-doubled every 5 years, hence realible aerodynamic computations can be anticipated in about (15-7) x 5 = 40 years

Thus, turbulence modelling is required

LES in wind energy

Why not (always) use RANS:





Basics of LES

Conceptual steps in LES:

(i) A *filtering* operation is defined to decompose the velocity $\boldsymbol{U}(\boldsymbol{x}, t)$ into the sum of a filtered (or resolved) component $\overline{\boldsymbol{U}}(\boldsymbol{x}, t)$ and a residual (or subgrid-scale, SGS) component $\boldsymbol{u}'(\boldsymbol{x}, t)$. The filtered velocity field $\overline{\boldsymbol{U}}(\boldsymbol{x}, t)$ – which is three-dimensional and time-dependent – represents the motion of the large eddies.

(ii) The equations for the evolution of the filtered velocity field are derived from the Navier–Stokes equations. These equations are of the standard form, with the momentum equation containing the *residual-stress tensor* (or SGS stress tensor) that arises from the residual motions.

(iii) Closure is obtained by modelling the residual-stress tensor, most simply by an eddy-viscosity model.

(iv) The model filtered equations are solved numerically for $\overline{\boldsymbol{U}}(\boldsymbol{x}, t)$, which provides an approximation to the large-scale motions in one realization of the turbulent flow.

Basics of LES

DNS and resolution in LES:

Table 13.1. *Resolution in DNS and in some variants of LES*

Model	Acronym	Resolution
Direct numerical simulation	DNS	Turbulent motions of all scales are fully resolved
Large-eddy simulation with near-wall resolution	LES-NWR	The filter and grid are sufficiently fine to resolve 80% of the energy everywhere
Large-eddy simulation with near-wall modelling	LES-NWM	The filter and grid are sufficiently fine to resolve 80% of the energy remote from the wall, but not in the near-wall region
Very-large-eddy simulation	VLES	The filter and grid are too coarse to resolve 80% of the energy

Filtering

General definition of filter:

$$\overline{U}(\boldsymbol{x}, t) = \int G(\boldsymbol{r}, \boldsymbol{x}) U(\boldsymbol{x} - \boldsymbol{r}, t)\, \mathrm{d}\boldsymbol{r},$$

where $$\int G(\boldsymbol{r}, \boldsymbol{x})\, \mathrm{d}\boldsymbol{r} = 1.$$

Residual field: $\boldsymbol{u}'(\boldsymbol{x}, t) \equiv \boldsymbol{U}(\boldsymbol{x}, t) - \overline{\boldsymbol{U}}(\boldsymbol{x}, t),$

Decomposition: $\boldsymbol{U}(\boldsymbol{x}, t) = \overline{\boldsymbol{U}}(\boldsymbol{x}, t) + \boldsymbol{u}'(\boldsymbol{x}, t)$

Remark that $\overline{\boldsymbol{u}'}(\boldsymbol{x}, t) \neq 0$

Features of filtering

Remark that $\overline{\boldsymbol{u}'}(\boldsymbol{x}, t) \neq 0$

Filtering and differentiation w.r.t. time commute:

$$\frac{\partial \overline{\boldsymbol{U}}}{\partial t} = \overline{\left(\frac{\partial \boldsymbol{U}}{\partial t}\right)}.$$

Filtering and taking means commute:

$$\overline{(\langle \boldsymbol{U} \rangle)} = \langle \overline{\boldsymbol{U}} \rangle.$$

Differentiation w.r.t. space gives:

$$\frac{\partial \overline{U}_i}{\partial x_j} = \overline{\left(\frac{\partial U_i}{\partial x_j}\right)} + \int U_i(\boldsymbol{x} - \boldsymbol{r}, t)\, \frac{\partial G(\boldsymbol{r}, \boldsymbol{x})}{\partial x_j}\, \mathrm{d}\boldsymbol{r}.$$

Examples of filters used in LES

Name	Filter function	Transfer function
General	$G(r)$	$\widehat{G}(\kappa) \equiv \int_{-\infty}^{\infty} e^{i\kappa r} G(r)\, \mathrm{d}r$
Box	$\frac{1}{\Delta} H(\frac{1}{2}\Delta - \lvert r \rvert)$	$\frac{\sin(\frac{1}{2}\kappa\Delta)}{\frac{1}{2}\kappa\Delta}$
Gaussian	$\left(\frac{6}{\pi\Delta^2}\right)^{1/2} \exp\left(-\frac{6r^2}{\Delta^2}\right)$	$\exp\left(-\frac{\kappa^2\Delta^2}{24}\right)$
Sharp spectral	$\frac{\sin(\pi r/\Delta)}{\pi r}$	$H(\kappa_c - \lvert\kappa\rvert)$, $\kappa_c \equiv \pi/\Delta$
Cauchy	$\frac{a}{\pi\Delta[(r/\Delta)^2 + a^2]}, \quad a = \frac{\pi}{24}$	$\exp(-a\Delta\lvert\kappa\rvert)$
Pao		$\exp\left(-\frac{\pi^{2/3}}{24}(\Delta\lvert\kappa\rvert)^{4/3}\right)$

Filtering in LES

Examples of 1-D filters:

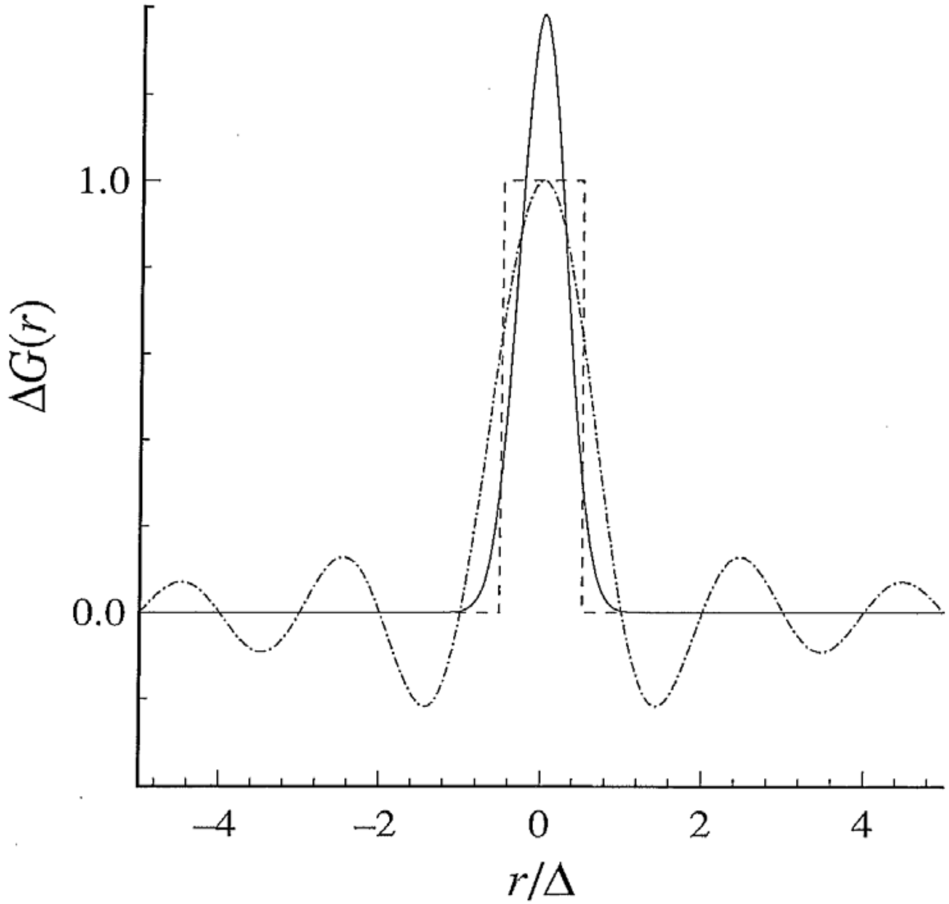


Fig. 13.1. Filters $G(r)$: box filter, dashed line; Gaussian filter, solid line; sharp spectral filter, dot–dashed line.

Filtering in LES

Example of 1-D filtering of velocity field:

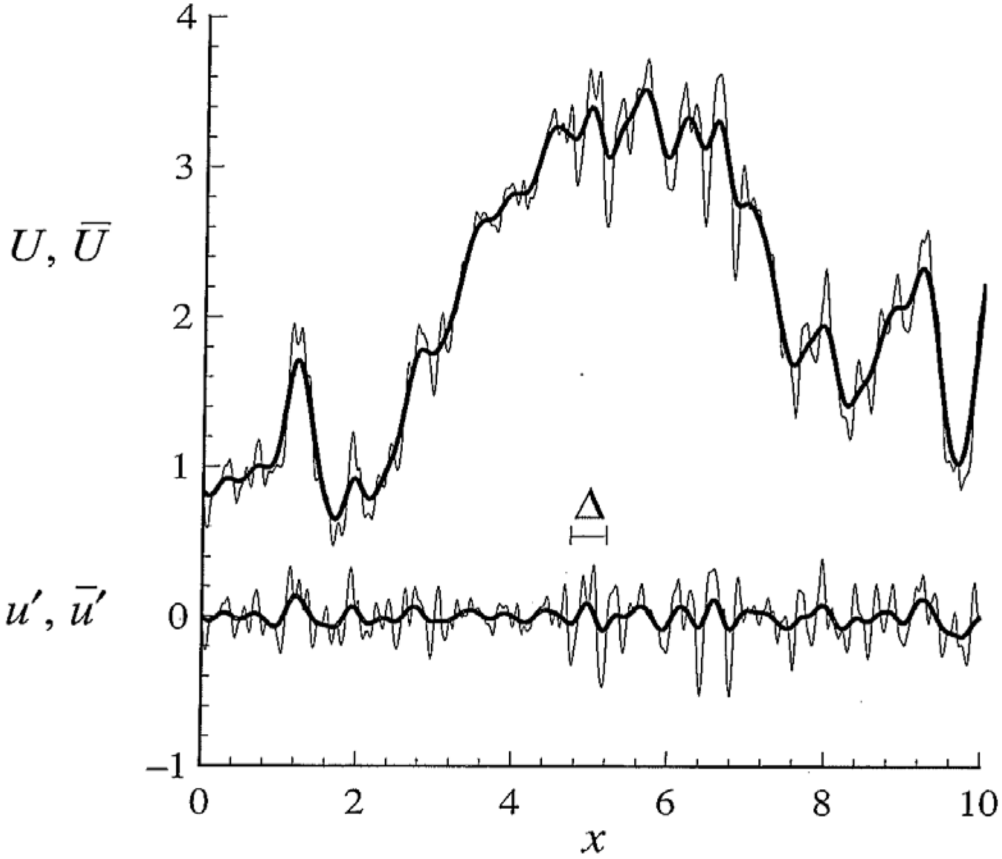


Fig. 13.2. Upper curves: a sample of the velocity field $U(x)$ and the corresponding filtered field $\overline{U}(x)$ (bold line), using the Gaussian filter with $\Delta \approx 0.35$. Lower curves: the residual field $u'(x)$ and the filtered residual field $\overline{u'(x)}$ (bold line).

Filtering in LES

The filtered energy spectrum:

Autocovariance: $R(r) \equiv \langle u(x+r)u(x) \rangle$

1-D spectrum: $$E_{11}(\kappa) = \frac{1}{\pi}\int_{-\infty}^{\infty} R(r)e^{-i\kappa r}\,\mathrm{d}r.$$

Filtered spectrum: $\overline{E}_{11}(\kappa) = |\widehat{G}(\kappa)|^2 E_{11}(\kappa)$

Transfer function: $$\widehat{G}(\kappa) \equiv \int_{-\infty}^{\infty} G(r)e^{-i\kappa r}\,\mathrm{d}r = 2\pi\mathcal{F}\{G(r)\}$$

Filtering in LES

Attenuation factors:

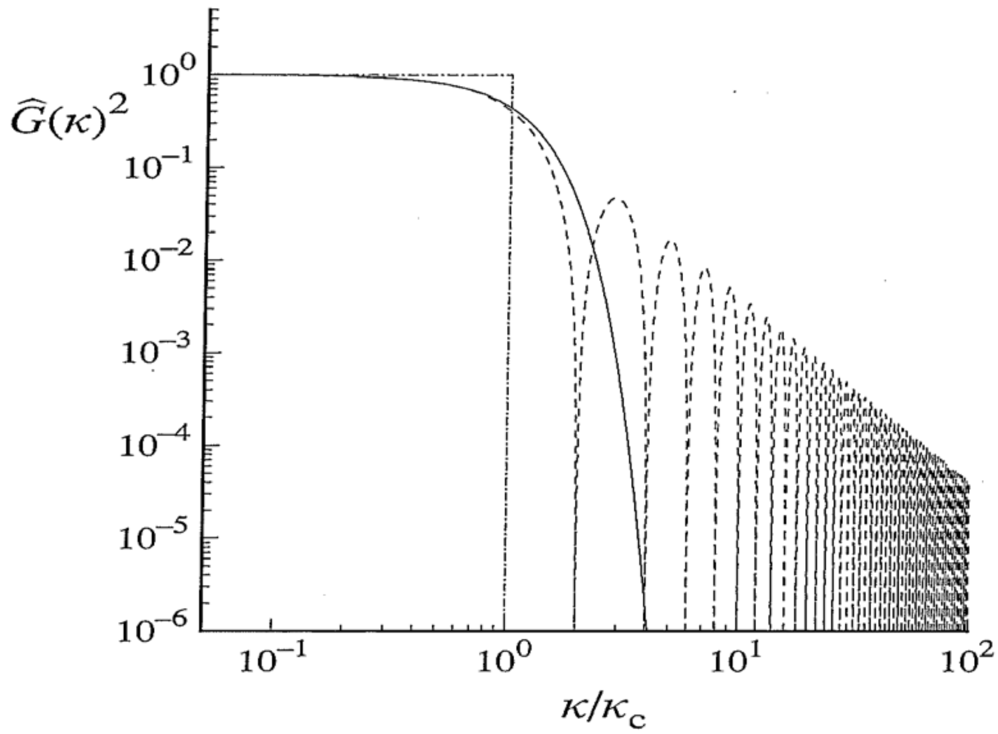


Fig. 13.4. Attenuation factors $\widehat{G}(\kappa)^2$: box filter, dashed line; Gaussian filter, solid line; sharp spectral filter, dot–dashed line.

Filtering in LES

Filtered energy spectrum:

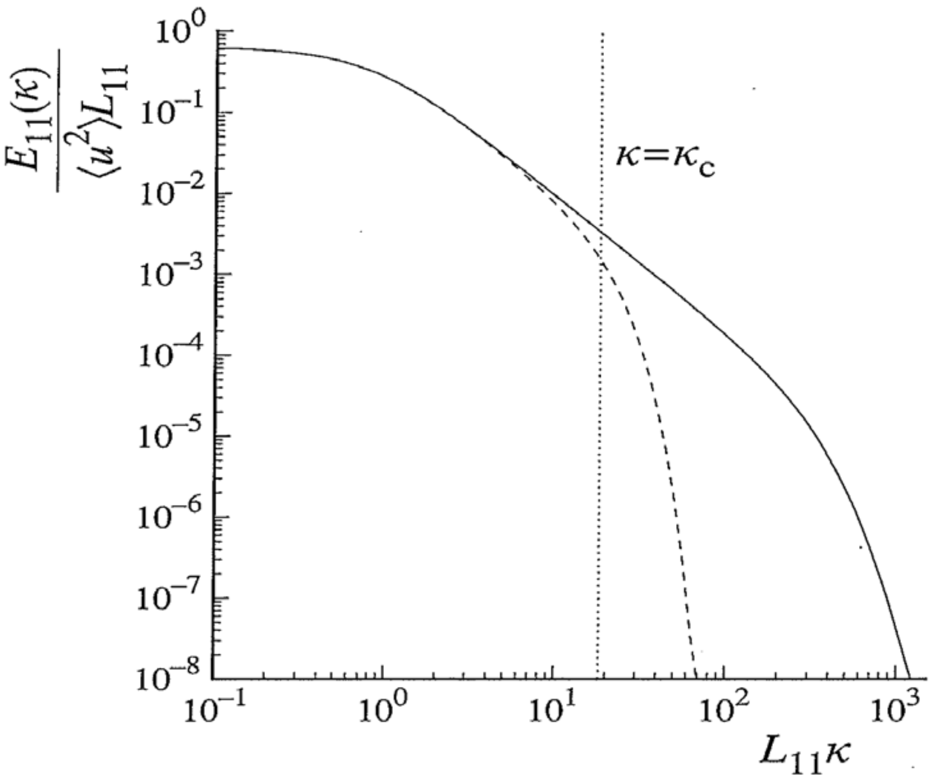


Fig. 13.5. The one-dimensional spectrum $E_{11}(\kappa)$ (solid line) obtained for the model spectrum at $R_\lambda = 500$; and the filtered spectrum $\overline{E}_{11}(\kappa)$ (dashed line) obtained using the Gaussian filter with $\Delta = \frac{1}{2}L_{11}$.

Filtering of conservation equations

Filtered continuity equation:

$$\overline{\left(\frac{\partial U_i}{\partial x_i}\right)} = \frac{\partial \overline{U}_i}{\partial x_i} = 0, \qquad \Longrightarrow \qquad \frac{\partial u'_i}{\partial x_i} = \frac{\partial}{\partial x_i}(U_i - \overline{U}_i) = 0.$$

Filtered momentum equation:

$$\frac{\partial \overline{U}_j}{\partial t} + \frac{\partial \overline{U_i U_j}}{\partial x_i} = \nu \frac{\partial^2 \overline{U}_j}{\partial x_i \partial x_i} - \frac{1}{\rho} \frac{\partial \overline{p}}{\partial x_j}.$$

Residual stress tensor: $\tau_{ij}^{\mathrm{R}} \equiv \overline{U_i U_j} - \overline{U}_i \overline{U}_j.$

Final form of filtered momentum equation:

$$\frac{\overline{\mathrm{D}}\, \overline{U}_j}{\overline{\mathrm{D}} t} = \nu \frac{\partial^2 \overline{U}_j}{\partial x_i \partial x_i} - \frac{\partial \tau_{ij}^{\mathrm{r}}}{\partial x_i} - \frac{1}{\rho} \frac{\partial \overline{p}}{\partial x_j}.$$

Residual stresses

Anisotropic residual stress tensor: $\tau^{\mathrm{r}}_{ij} \equiv \tau^{\mathrm{R}}_{ij} - \frac{2}{3} k_{\mathrm{r}} \delta_{ij}.$

Residual kinetic energy: $k_{\mathrm{r}} \equiv \frac{1}{2} \tau^{\mathrm{R}}_{ii}$

Decomposition: $\overline{U_i U_j} = \overline{U}_i \overline{U}_j + \tau^{\mathrm{R}}_{ij}$

$$\tau^{\mathrm{R}}_{ij} = \mathcal{L}^{\mathrm{o}}_{ij} + \mathcal{C}^{\mathrm{o}}_{ij} + \mathcal{R}^{\mathrm{o}}_{ij}$$

Leonard stresses: $\mathcal{L}^{\mathrm{o}}_{ij} \equiv \overline{\overline{U}_i \overline{U}_j} - \overline{\overline{U}}_i \overline{\overline{U}}_j$

Cross stresses: $\mathcal{C}^{\mathrm{o}}_{ij} \equiv \overline{\overline{U}_i u'_j} + \overline{u'_i \overline{U}_j} - \overline{\overline{U}}_i \overline{u'_j} - \overline{u'_i}\, \overline{\overline{U}}_j$

SGS Reynolds stresses: $\mathcal{R}^{\mathrm{o}}_{ij} \equiv \overline{u'_i u'_j} - \overline{u'_i}\, \overline{u'_j}$

The Smagorinsky model

Eddy-viscosity model: $\tau_{ij}^{\mathrm{r}} = -2\nu_{\mathrm{r}}\overline{S}_{ij}$

Mixing length anzats: $\nu_{\mathrm{r}} = \ell_{\mathrm{S}}^2 \overline{\mathcal{S}} = (C_{\mathrm{S}}\Delta)^2 \overline{\mathcal{S}}$

where $\overline{S}_{ij} \equiv \frac{1}{2}\left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i}\right)$ **and** $\overline{\mathcal{S}} \equiv (2\overline{S}_{ij}\overline{S}_{ij})^{1/2}$

Transfer of energy to residual motions:

$$\mathcal{P}_{\mathrm{r}} \equiv -\tau_{ij}^{\mathrm{r}}\overline{S}_{ij} = 2\nu_{\mathrm{r}}\overline{S}_{ij}\overline{S}_{ij} = \nu_{\mathrm{r}}\overline{\mathcal{S}}^2$$

The Smagorinsky constant

Behaviour in the inertial subrange:

$$\varepsilon = \langle \mathcal{P}_r \rangle = \langle \nu_r \overline{\mathcal{S}}^2 \rangle = \ell_S^2 \langle \overline{\mathcal{S}}^3 \rangle$$

Assume Kolmogorov spectrum:

$$\langle \overline{\mathcal{S}}^2 \rangle \approx 2 \int_0^\infty \kappa^2 \widehat{G}(\kappa)^2 C \varepsilon^{2/3} \kappa^{-5/3} \, d\kappa = a_f C \varepsilon^{2/3} \Delta^{-4/3}$$

where $a_f \equiv 2 \int_0^\infty (\kappa\Delta)^{1/3} \widehat{G}(\kappa)^2 \Delta \, d\kappa$

Assuming a sharp spectral filter: $a_f = \frac{3}{2}\pi^{4/3}$

We get: $\ell_S = \frac{\Delta}{(C a_f)^{3/4}} \left(\frac{\langle \overline{\mathcal{S}}^3 \rangle}{\langle \overline{\mathcal{S}}^2 \rangle^{3/2}} \right)^{-1/2}$ **or:** $C_S = \frac{\ell_S}{\Delta} = \frac{1}{\pi}\left(\frac{2}{3C}\right)^{3/4} \approx 0.17$

The Smagorinsky filter

Filter transfer function: $\widehat{G}(\kappa) = \left(\frac{\overline{E}(\kappa)}{E(\kappa)}\right)^{1/2}$

Assume e.g. Pao's model spectrum:

$$E(\kappa) = f_L(\kappa L) C \varepsilon^{2/3} \kappa^{-5/3} \exp\left[-\tfrac{3}{2} C (\kappa\eta)^{4/3}\right]$$

We get $\widehat{G}(\kappa) = \exp[-\tfrac{3}{4} C \kappa^{4/3} (\overline{\eta}^{4/3} - \eta^{4/3})]$

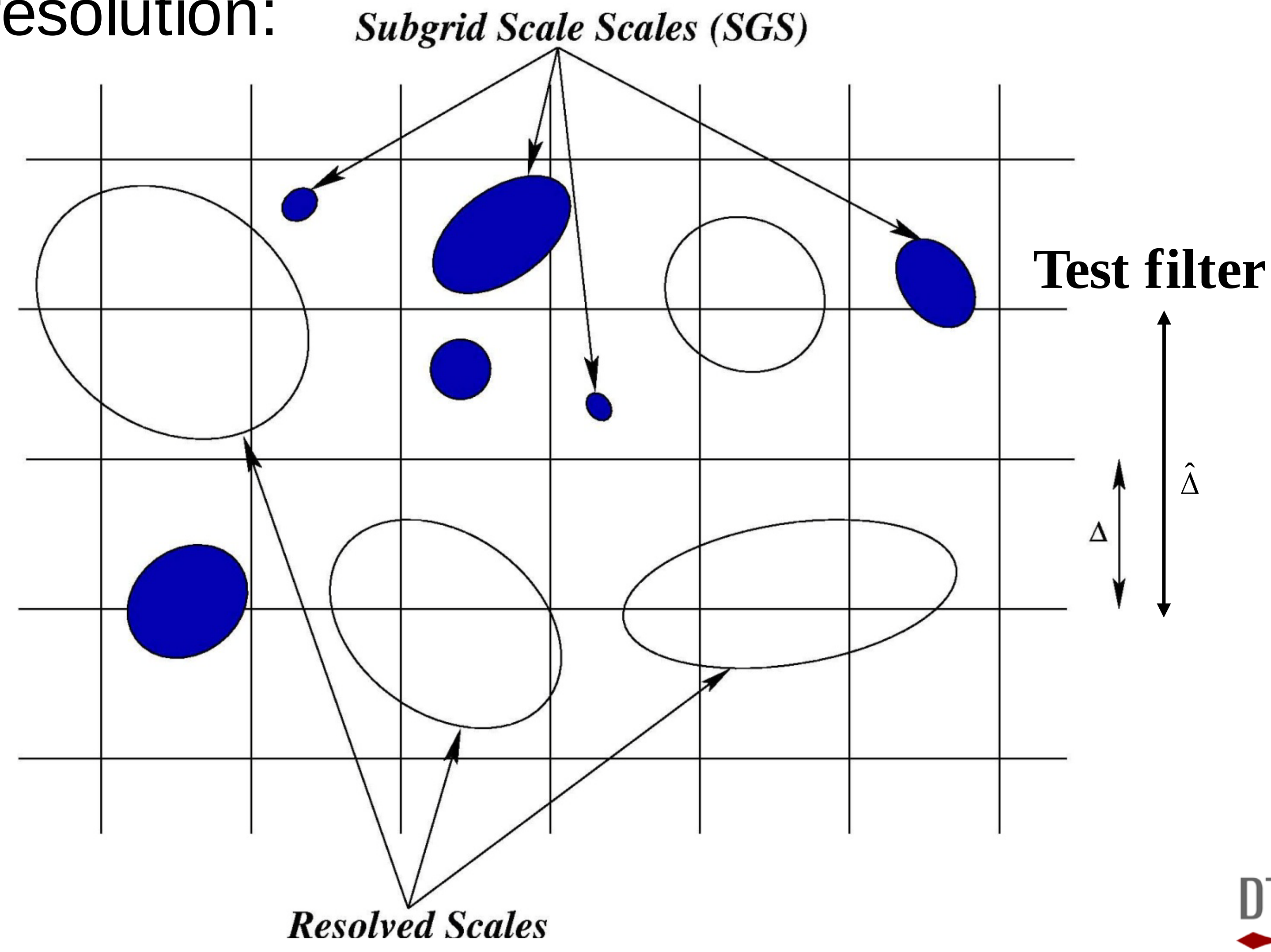
where $\overline{\eta} \equiv \left(\frac{(\nu + \nu_r)^3}{\varepsilon}\right)^{1/4} = \ell_S \left(1 + \frac{\nu}{\nu_r}\right)^{1/2}$

Smagorinsky constant:

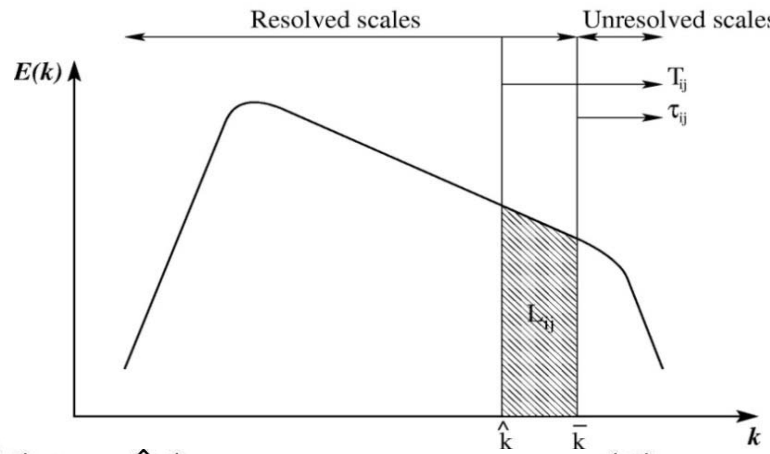
$$C_S = \frac{\pi^{1/2}}{(18C)^{3/4}} \left[1 + \frac{18C}{\pi^{2/3}} \left(\frac{\eta}{\Delta}\right)^{4/3}\right]^{1/4} \approx 0.15 \left[1 + \left(\frac{7\eta}{\Delta}\right)^{4/3}\right]^{1/4}$$

LES and turbulence

Flow resolution:



Mixed scale model:

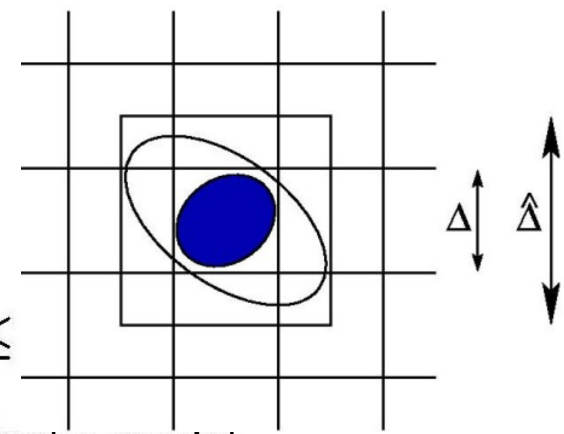


$$k_{sgs} = \frac{1}{2}\overline{u_i'}\,\overline{u_i'}$$

$$\overline{u_i'} = \bar{u}_i'$$

$$k_{sgs} = \frac{1}{2}\overline{u_i'}\,\overline{u_i'} = (\bar{u}_i - \hat{\bar{u}}_i)(\bar{u}_i - \hat{\bar{u}}_i) \qquad (6)$$

$$\hat{\bar{u}}_i = \frac{1}{4}(\bar{u}_{i-1} + 2\bar{u}_i + \bar{u}_{i+1})$$



$$\nu_{sgs} = C_m \Delta^{1+\alpha} k_{sgs}^{\frac{1+\alpha}{2}} \left|\bar{S}_{ij}\right|^{\alpha}$$

$$C_m = C_q^{1-\alpha} C_s^{2\alpha},\ C_q \approx 0.20,\ 0 \le \alpha \le$$

$\alpha = 1$ equation transforms to equation from Smagorinsky model.