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1 Linear motion of spar turbine

A moored spar buoy may be used as support of an offshore wind turbine. The buoy is a slender vertical circular cylinder of large draft. It is moored with slack mooring lines. That they are slack means that there role is merely to keep the wind turbine in position rather than restricting its linear reponses to the wave forces and wind gusts. A spar buoy is used to support the Hywind wind turbine.

1.1 Example. Data of Statoil's Hywind, source: Internet

2.3 MW turbine size
138 tons, turbine weight
65 m, turbine height
82.4 m, rotor diameter
100 m, hull draft
5300 m^3, displacement
6 m, diameter water line
8.3 m, diameter, submerged body
100-170 m, water depths

1.2 Wave and wind loads

The incoming wave field imposes an exciting force acting on the wetted surface of the floating spar buoy. This force can be modelled by

$$F^{wave} = \mathrm{Re}[\sum_n] A_n X^{wave}(\omega_n, \theta_n) e^{\mathrm{i}\omega_n t}] = \mathrm{Re} \int\int dA X^{wave}(\omega, \theta) e^{\mathrm{i}\omega t}] \qquad (1)$$

The wind force F^{wind} is discussed elsewhere in the course. The sum of the wave and wind force acting on the spar buoy and the turbine results in motions of the structure.

1.3 Analysis in the frequency plane

We first analyze the forces imposed by the incoming waves, diffracted and radiated waves, including mass and added mass effects. The analysis is best carried out in the frequency plane, where the forces and motions are analyzed at the individual frequency. The complete motion is analyzed in the end, where we return to from the frequency domain to the time domain.

It is convenient to introduce a frame of reference which is fixed in space, where (x_1, x_2) are in the horizontal plane and y vertical upward. We will sometimes use $x_1 = x$ and $x_2 = z$.

The wave and wind excitation are considered in the frequency plane where also the body responses are evaluated. The wave force reads

$$F_1^{wave} = \mathrm{Re}(X_1^{wave} e^{\mathrm{i}\omega t}) \qquad (2)$$

X_1^{wave} denotes the wave exciting force due to a wave of frequency component ω and amplitude A and Re means real part. In a linear analysis the wave force given in (2) acts on the body in its mean position. The force acts on the symmetrical circular cylinder along the wave propagation direction.

The linear wave exciting force may be given by the McCamy-Fuchs solution. In the case of a slender cylinder it may be given approximately from the inertia term in Morison's equation which gives that $X_1^{wave} = 2\pi\rho g R^2 A$ where a complex phase is included in the amplitude A. In the slender cylinder range the two estimates of the force are numerically very close. The approaches are discussed elsewhere in the course.

We assume that the body is exposed to wind and waves moving along the positive x-direction. Further, the body response is composed by oscillations along the x-direction (surge) and about the x_2-axis (pitch) given by

$$\hat{\xi}_1(t) = \mathrm{Re}(\xi_1 e^{\mathrm{i}\omega t}), \qquad \hat{\xi}_6(t) = \mathrm{Re}(\xi_6 e^{\mathrm{i}\omega t}).$$

1.4 Equation of motion. Wave and hydrodynamic effects

Conservation of momentum along the x-axis gives

$$[-\omega^2(M_{11} + a_{11}) + \mathrm{i}\omega b_{11} + c_{11}]\xi_1 + [-\omega^2(M_{16} + a_{16}) + \mathrm{i}\omega b_{16} + c_{16}]\xi_6 = X_1^{wave} + X_1^{wind}. \quad (3)$$

Similarly, conservation of angular momentum with respect to the x_2-axis gives

$$[-\omega^2(M_{61} + a_{61}) + \mathrm{i}\omega b_{61} + c_{61}]\xi_1 + [-\omega^2(M_{66} + a_{66}) + \mathrm{i}\omega b_{66} + c_{66}]\xi_6 = X_6^{wave} + X_6^{wind}. \quad (4)$$

In (4) X_6^{wave} and X_6^{wind} are moments due to the waves and the wind. In (3) and (4) terms represent

- M_{11} total mass $m = \rho V$ of the floating structure
- a_{11} added mass of the spar buoy, in surge
- $M_{66} = I_{66}$ moment of inertia
- a_{66} added moment of inertia
- b_{11}, b_{66} coefficients determining the damping due to waves and wind
- c_{11} restoring force coefficient in the horizontal direction (moorings)
- $c_{66} = \rho g V[(S_{11}/V) + y_b - y_g] + c_{66}^{moorings} \simeq mg[y_b - y_g] + c_{66}^{moorings}$ restoring moment
- $M_{16} = M_{61}, a_{16} = a_{61}, b_{16} = b_{61}, c_{16} = c_{61}$ cross coupling coefficients

In the equations above $M_{11} = m = \rho V$ is the mass of the structure which equals the displaced water mass given by ρV, ρ density of the water and V submerged volume.

a_{11} denotes added mass corresponding to the inertia force of the accelerated fluid due to the oscillations in surge. A (horizontal) section of the spar buoy has an added mass of $\rho\pi R^2$ where R the radius, assuming a strip theory approach. Assuming constant radius of the spar buoy and that h_2 is its draft, the added mass of the submerged volume becomes $a_{11} = \rho\pi R^2 h_2 = \rho V = m$.

b_{11} denotes the damping coefficient of the surge motion. For pure wave radiation this represents a damping force due to the outward going waves caused by the body motion. The pure wave damping coefficient is very small since the sylinder is slender ($kR << 1$). It may be shown that with only wave radiation $\omega b_{11} = \pi^2 \rho g k^2 R^4$. This is very small compared to the mass plus added mass terms and is not essential in the analysis.

b_{66} denotes the damping coefficient of the yaw motion. Its contribution from the waves is small but positive.

The damping coeffients due to the coupling between the body responses and the wind force, which are negative damping forces, have been analyzed above.

The mass coupling term is $M_{16} = -my_g$ where y_g denotes the center of gravity. $a_{16} \simeq -\pi\rho R^2 \frac{1}{2} h_2^2 = -my_b$ denotes the added mass cross coupling term between modes 1 and 6 and $y_b = -\frac{1}{2}h_2$ denotes the vertical coordinate of the buoyancy center of the submerged part of the structure. It is convenient to define the vertical position $y_0 = \frac{1}{2}(y_g + y_b)$. The sum $M_{16} + a_{16}$ then becomes $-2my_0$.

The restoring force coefficient c_{11} is caused by the mooring lines which are slack and which introduce forces of magnitude comparable to the second order in the incoming wave amplitude. This means that c_{11} is much smaller than the inertia terms in the equation and is neglected in the present analysis.

The restoring force due to buoyancy and gravity forces are, in the yaw mode of motion,

$$c_{66} \simeq mg(y_b - y_g),$$

where y_b and y_g denote the vertical positions of the center of bouyancy of the spar buoy and center of gravity of the structure, respectively. The distance $y_b - y_g$ denotes the metacentric height and is positive. This difference is usually a small fraction of the draft (h_2) of the spar, i.e. $y_b - y_g = \beta_1 h_2$ where β_1 is typically 0.1-0.2. (The usual waterplane moment in the metacentric height is small for a slender vertical cylinder of deep draft and is omitted.

1.5 Example. Resonant periods of Hywind

At resonance the restoring and inertia forces are in balance, giving in surge

$$\omega_0^2 = \frac{c_{11}}{m + a_{11}}.$$

The resonance of Hywind in surge/sway is 120 seconds corresponding to a resonance frequency of $\omega_0 = 0.052 \text{ s}^{-1}$. This determines the horizontal restoring force coefficient by $c_{11} = \omega_0^2(m + a_{11})m = 5.4 \cdot 10^{-3} \cdot m$ with dimension m^{-1}.

Resonance in yaw. We assume that the effect of the moorings are much weaker than the effects of buoyancy and gravity, which is reasonable, referring to the low resonance frequency in surge/sway. Neglecting the mooring force, the resonance frequency in the yaw mode of motion is then approximated by

$$\omega_0^2 = \frac{mg(y_b - y_g)}{I_{66} + a_{66}}.$$

As above $y_b - y_g = \beta_1 h_2$. The sum of the moment of inertia and added moment of inertia may be written by $\beta_2 m h_2^2$ where β_2 is typically in the range 0.2-0.3. The resonance frequency then becomes $\omega_0^2 = \beta_1 g / \beta_2 h_2$.

With $\beta_1 = 0.1$ and $\beta_2 = 0.2$ the resonance frequency squared becomes $\omega_0^2 = 0.05\mathrm{s}^{-2}$ and resonance period $T = 28$ sec.

With $\beta_1 = 0.2$ and $\beta_2 = 0.3$ the resonance frequency squared becomes $\omega_0^2 = 0.067\mathrm{s}^{-2}$ and resonance period $T = 24$ sec.

For comparison, the resonance period in yaw of Hywind is 25 sec.

1.6 Evaluation of the responses

We consider the equations of motion where the moments are evaluated with respect to the vertical position $y_0 = \frac{1}{2}(y_b + y_g)$. We disregard for the moment the effects of c_{11}, b_{16} and c_{16}. For surge and yaw we get, approximately, with the wind excitation included below,

$$-2m\omega^2\xi_1 = \frac{2mg}{h_2}A \tag{5}$$

$$[-\omega^2(M_{66} + a_{66}) + mg(y_b - y_g)]\xi_6 = AX_6^{wave}. \tag{6}$$

where

$$X_6^{wave} = 2\pi\rho g R^2 A\Big(\frac{1}{k} + y_0\Big) \tag{7}$$

Multiplying the equations by $e^{\mathrm{i}\omega t}$ and summing over all frequencies and wave angles, the equations may alternatively be obtained in the time domain. Including the wind force we find

$$2m\ddot{\xi}_1 = \frac{2mg}{h_2}\eta(t) + F^{wind}(t) \tag{8}$$

$$\beta_2 m h_2^2 \ddot{\xi}_6 + mg\beta_1 h_2 \xi_6 = \frac{2mg}{h_2}\Big(y_0\eta(t) + \mathrm{Re}\sum_n \frac{A_n}{k_n}e^{\mathrm{i}\omega_n t}\Big) + M_6^{wind}(t) \tag{9}$$

where $|A_n| = (2S(\omega_n)\Delta\omega_n)^{\frac{1}{2}}$, $0 < arg(A_n) < 2\pi$ and $M_6^{wind}(t)$ denotes the moment with respect to y_0 due to the wind. In the equation above we have also used $M_{66} + a_{66} = \beta_2 m h_2^2$ and $mg(y_b - y_g) = mg\beta_1 h_2$.

1.7 Example

Assume that the incoming waves are swells and that the wind is smooth with weak gusts. Then

$$\frac{\xi_1}{A} \simeq -\frac{1}{kh_2} = -\frac{\lambda}{2\pi h_2}$$
$$\frac{\xi_6 Y}{A} \simeq \frac{Y/h_2}{\beta_2(-1 + \omega_0^2/\omega^2)}\frac{\lambda}{\pi h_2}\Big(\frac{\lambda}{2\pi h_2} + \frac{y_0}{h_2}\Big)$$

If $\lambda/\pi h_2 \sim 1$, $\beta_1 = 0.2$, $\beta_2 = 0.3$ we obtain $\xi_1/A \sim -\frac{1}{2}$ and $\xi_6 Y/A \sim 0.6$, which means that the horizontal excursions of the wind turbine is about the same as the wave amplitude.

1.8 Responses in irregular waves and wind gusts

We may assume that the variables representing the wave field and wind gusts are independent stochastic variables, each of them with zero mean and non-zero variance, but with zero cross coupling. The body responses are then splitted into two parts, each of them either caused by the wave field or the wind gusts. Let

$$\hat{\xi}_1(t) = \hat{\xi}_1^{wave}(t) + \hat{\xi}_1^{wind}(t) \tag{10}$$

and similarly for the response in the yaw mode of motion. For the wave part we obtain by linear superposition

$$\hat{\xi}_i^{wave}(t) = \mathrm{Re} \int \int (\xi_i^{wave}/A)(\omega, \theta) e^{\mathrm{i}\omega t} dA \tag{11}$$

and similarly for the wind part which is discussed elsewhere in the course.

By evaluation of the expectation of the variables squared we obtain

$$E[(\hat{\xi}_i^{wave})^2] = \int_0^\infty \int_0^{2\pi} S_{wave}(\omega, \theta) |(\xi_i^{wave}/A)(\omega, \theta)|^2 d\omega d\theta \tag{12}$$

where $S_{wave}(\omega, \theta)$ denotes the wave spectrum and $(\xi_i^{wave}/A)(\omega, \theta)$ response amplitude operator.

Similar result is obtained for the wind induced motion and is obtained elsewhere in the course.