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1 Wave force in strongly nonlinear periodic waves

1.1 Force at the fundamental wave period

A fixed vertical circular cylinder of deep draft exposed to periodic waves experiences a horizontal wave force. The dominant part of this force oscillates according to the period T of the wave - the fundamental period. We examine the properties of the oscillatory force at the fundamental period, for a range of steepness of the incoming waves.

The surface elevation η of Stokes waves in deep water is, to leading order in the wave amplitude, given by

$$\eta = A\cos(-kx + \omega t),$$

where $\omega = 2\pi/T$ denotes the frequency and k the wavenumber. The x-axis is chosen along the wave direction and t denotes time. The nonlinear extension of the surface elevation can be obtained explicitly by $\eta = A\cos\chi + \frac{1}{2}A^2k\cos 2\chi + \frac{3}{8}A^3k^2\cos 3\chi + ...$ with $\chi = -kx + \omega t$.

The frequency ω is related to the wavenumber k by the nonlinear dispersion relation which reads, in deep water

$$\omega^2 = gk(1 + (Ak)^2) \tag{1}$$

The corresponding velocity potential is given by

$$\phi = -\frac{gA}{\omega}e^{ky}\sin(-kx + \omega t) \tag{2}$$

and is correct including cubic nonlinear terms. The vertical axis y points upwards and $y = 0$ determines the water surface at rest. The wave induced velocity is deduced from the velocity potential. The horizontal component $u = \partial\phi/\partial x$ along the wave direction becomes

$$u(x, y, t) = \frac{gAk}{\omega}e^{ky}\cos(-kx + \omega t) \tag{3}$$

With such a wave input, Huseby and Grue (2000) measured the horizontal wave force on a vertical cylinder, in wave tank. The wave slope Ak was varied from zero to 0.24, where the latter implies a rather strongly nonlinear periodic wave train.

1.2 Calculation of the linear wave force

Consider the linear wave force acting on a slender vertical circular cylinder. That the wave is linear means that $Ak << 1$. That the cylinder is slender means that the radius is small compared to the wave length, such that the product between the wavenumber k and radius R is small ($kR << 1$).

The force may in this limit be evaluated by using the Morison equation section wise along the vertical, i.e.

$$\frac{dF}{dy} = \pi\rho R^2 C_m \dot{U}. \tag{4}$$

Here $\dot{U}$ denotes the horizontal acceleration of the wave at the central position of the cylinder ($x = 0$) and $C_m = 2$ is the mass coefficient in the inviscid limit. By vertical integration assuming a cylinder of deep draft, the force becomes

$$F(t) = 2\pi\rho g A R^2 \sin\omega t. \tag{5}$$

1.3 McCamy-Fuchs solution

Alternatively, the linear exciting force on the vertical cylinder may be obtained by the McCamy-Fuchs (1954) wave diffraction solution which has no restriction on the wavenumber. The wave potential is obtained by

$$\phi(x_1, x_2, y, t) = \mathrm{Re}\Big[\frac{\mathrm{i}g}{\omega}\eta(x_1, x_2, y)\frac{\cosh k(y+h)}{\cosh kh}e^{\mathrm{i}\omega t}\Big] \tag{6}$$

where η is the free surface displacement and satisfies the two-dimensional Helmholtz equation. The elevation of the incoming wave is given by $\eta_{inc} = Ae^{-\mathrm{i}kx_1}$. By using that

$$e^{-\mathrm{i}kx_1} = \sum_{m=1}^{\infty} \epsilon_m (\mathrm{i})^{-m} J_m(kr)\cos m\theta \tag{7}$$

where $(x_1, x_2) = r(\cos\theta, \sin\theta)$, J_m denotes Bessel function of first kind of order m, $\epsilon_0 = 1$ and $\epsilon_m = 2$ ($m > 0$). Taking into account the kinematic boundary condition at the fixed vertical cylinder, at $r = R$, the elevation η of the combined incoming and diffracted wave field reads

$$\eta(x_1, x_2) = A\sum_{m=1}^{\infty} \epsilon_m (\mathrm{i})^{-m}(J_m(kr) - H_m^{(2)}(kr)\frac{J_m'(kR)}{H_m^{(2)\prime}(kR)}\Big)\cos m\theta, \tag{8}$$

where $H_m^{(2)}$ denotes Hankel function of order m of the second kind, and a prime means derivative. Integrating the pressure along the θ-direction and along the vertical, we obtain the following expression for the linear wave force

$$F_1 = \frac{4\rho g A \tanh kh}{k^2 H_1^{(2)\prime}(kR)} \tag{9}$$

where the time dependent force is obtained by $F(t) = \mathrm{Re}[F_1 e^{\mathrm{i}\omega t}]$. In (9) $H_1^{(2)}$ denotes the Hankel function of second kind and order one, and a prime differentiation.

1.4 Force at the fundamental frequency in the nonlinear regime

Expressions (5) and (9) show that the linear force scales accoring to $\rho g A R^2$. Few calculations of the force oscillating at the fundamental frequency exist, when the waves are in the nonlinear regime. The difficulty with such calculations is that the waves exibit local breaking

at the waterline of the cylinder, when the waves are sufficiently strong. This local breaking is difficult to circumvent. Ferrant (1998) was able to perform calculations of the nonlinear force for waveslope Ak up to 0.145. Huseby and Grue (2000) performed laboratory measurements of the wave force for Ak up to 0.24, well into the regime where local breaking occurs at the cylinder. The waves were in the long wave regime with kR in the range 0.166-0.378. The measurements show that the nondimensional force is (about) constant through the wave slope range (figures 1-2). It is rather close to the values that can be obtained from the linear forces (5) and (9), for the whole range. Both force amplitude and phase are (about) constant for the range. For example, for $kR = 0.166$ we have that $|F_1|/\rho g A R^2$ is

2π	Morison's eq. (linear)
6.4	McCamy-Fuchs (linear)
6.45 ± 0.05	experiments, $R = 4$ cm (nonlinear)
6.65 ± 0.05	experiments, $R = 3$ cm (nonlinear)

1.5 Example

Offshore wind turbines may be exposed to trains of rather strong amplitude. A typical amplitude is 5 m. A typical period is 10 s. This implies a wavelength of $\lambda = 160$ m (deep water) corresponding to a wavenumber of $k = 2\pi/\lambda \simeq 0.04$ m^{-1}. With a cylinder diameter of 6 m, the cylinder radius R times the wavenumber is then $kR = 0.12$. The waveslope Ak becomes 0.2.

1.6 Exercise

Derive the expression for the force F in (5) using (4) for the sectionwise force with the acceleration derived from the orbital velocity in (3).

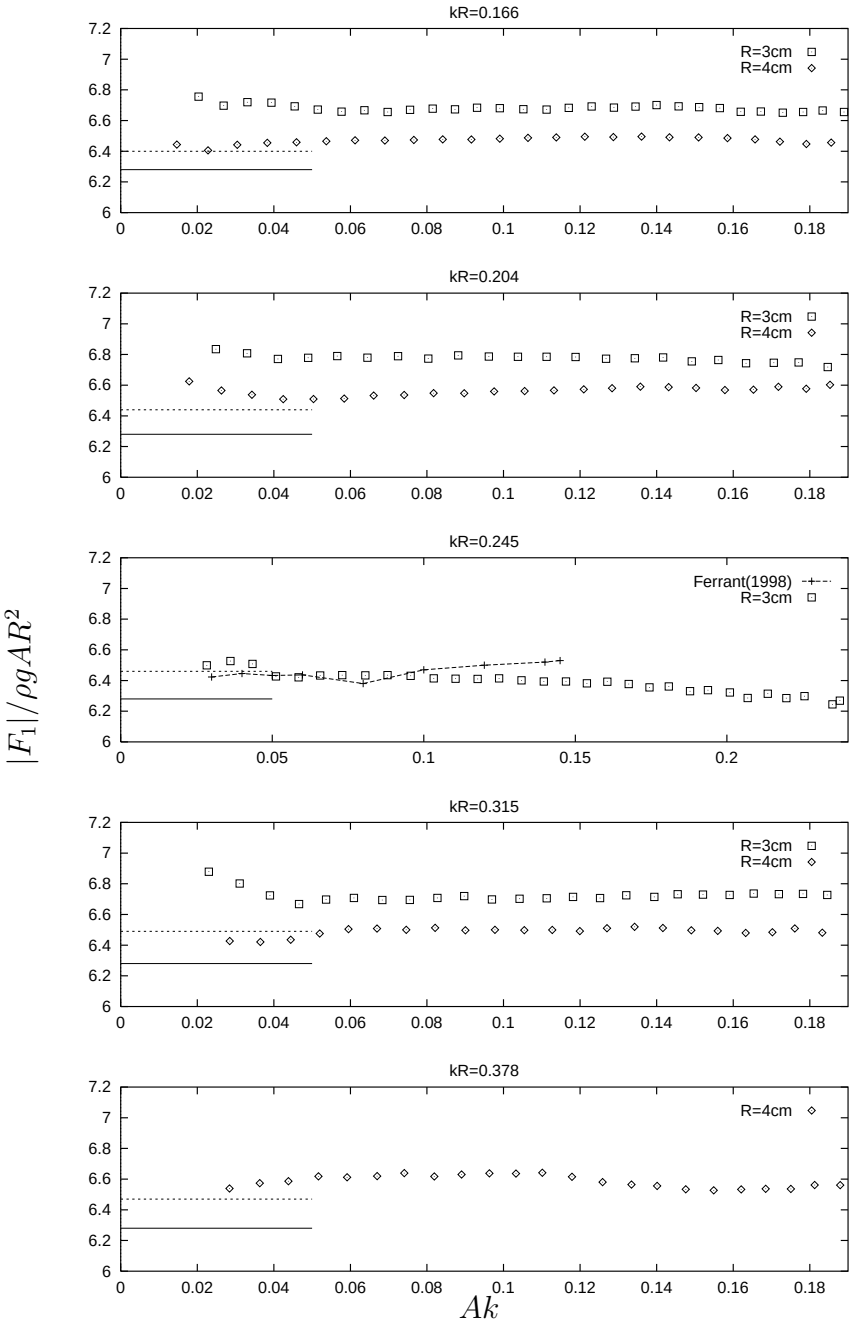

Figure 1: First harmonic force. Measurements by Huseby and Grue (2000) (squares and diamonds), Morison's equation (solid line), McCamy-Fuchs solution (dashed line), nonlinear computations by Ferrant (1998) for $kR = 0.245$ (dashed line with crosses). From Huseby and Grue (2000).

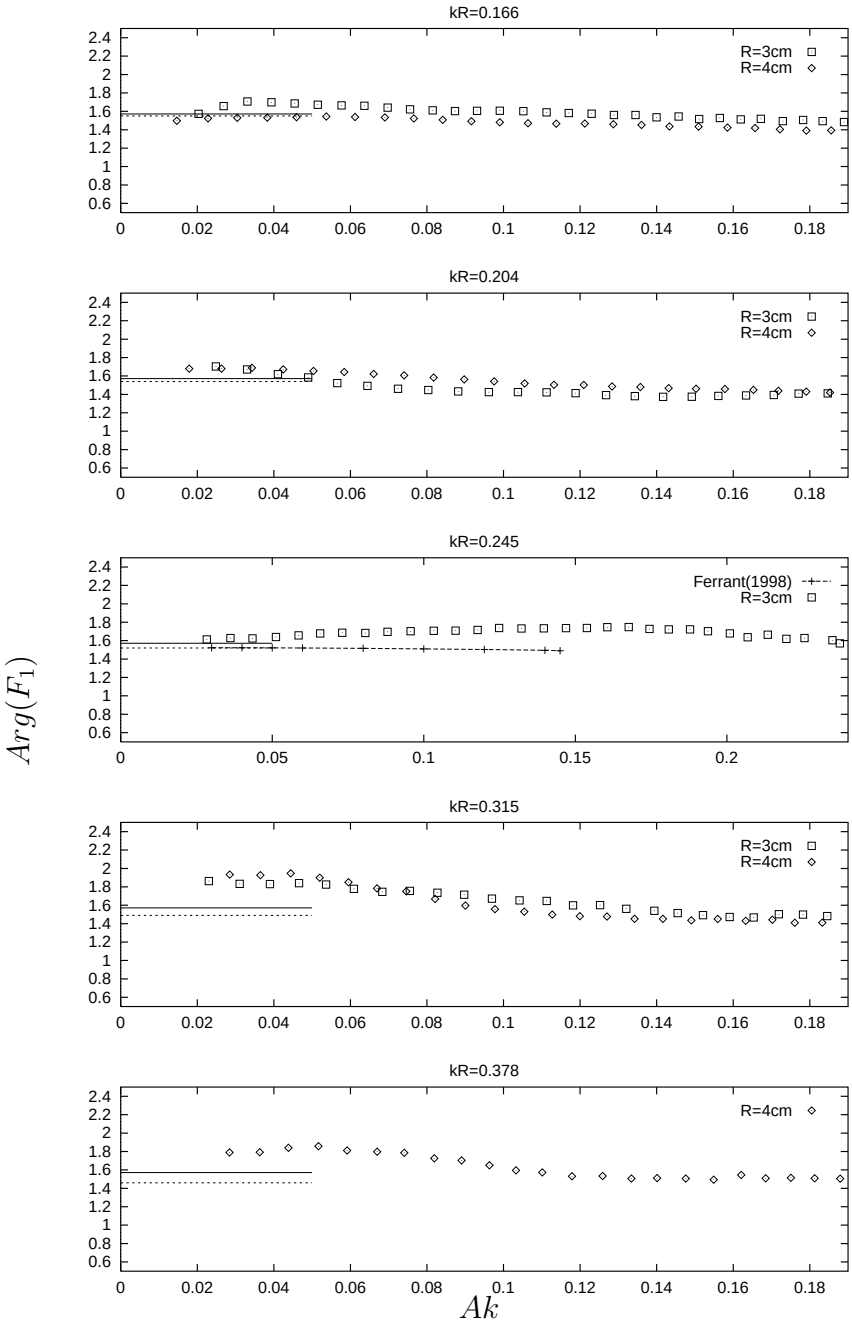

Figure 2: Phase of first harmonic force, $Arg(F_1)$ vs. Ak. Measurements by Huseby and Grue (2000) (squares and diamonds), Morison's equation (solid line), McCamy-Fuchs solution (dashed line), nonlinear computations by Ferrant (1998) for $kR = 0.245$ (dashed line with crosses). From Huseby and Grue (2000).

1.7 References

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