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1 Wave environment

Analysis of ocean surface waves, and more so, applied analysis of ocean surface waves, commonly starts out by a linear analysis of the waves. When we study the behaviour of fixed intallations or floating objects at sea, the first analysis one performs, is a linear analysis. This has the power of revealing the leading order behaviour of the structures and systems in consideration.

The surface elevation $\eta(t; x_1, x_2)$ at a horizontal position $\mathbf{x} = (x_1, x_2)$ of the ocean may be recorded in several ways providing a time series of the data that can be analyzed. Alternatively, snapshots of patches of the surface elevation provide $\eta(x_1, x_2; t)$. In a linear reconstruction of the wave elevation we assume that it is composed by a sum of linear waves of complex amplitude a_n, frequency ω_n and wavenumber vector $\mathbf{k}_n = (k_1, k_2)_n$ giving

$$\eta(x_1, x_2, t) = \mathrm{Re} \sum_{n=1}^{N} A_n e^{\mathrm{i}(-\mathbf{k}_n \cdot \mathbf{x} + \omega_n t)}, \tag{1}$$

where N denotes the number of components.

Processes like the one represented in (1) may be interpreted and analyzed in a statistical sense and are described in many text books.

An alternative to the sum in (1) is an integral representation by

$$\eta(x_1, x_2, t) = \mathrm{Re} \int \int dA(\omega, \theta) e^{\mathrm{i}(-\mathbf{k}(\omega) \cdot \mathbf{x} + \omega t)}, \tag{2}$$

The complex wave amplitudes in (1) are represented by the wave spectra that can be extracted from the statistical wave analysis. The complex amplitudes A_n have a real amplitude $|A_n|$ and phase $-\delta_n = arg(A_n)$ where the minus sign is chosen for convenience. The amplitudes $|A_n|$ are obtained from the wave spectrum by

$$\tfrac{1}{2}|A_n|^2 = S(\omega_n)\Delta\omega_n. \tag{3}$$

Moments of the wave spectrum are given by

$$m_k = \int_0^\infty \omega^k S(\omega) d\omega \tag{4}$$

The peak period of the spectrum is indicated by T_0. The mean wave period of the spectrum is defined by

$$T_2 = 2\pi (m_0/m_2)^{\frac{1}{2}}.$$

Another wave period commonly used is defined by

$$T_1 = 2\pi m_0/m_1.$$

A significant wave height, defined as the mean of the one highest waves, is defined by

$$H_s = 4m_0^{\frac{1}{2}}$$

An example of engineering spectrum is the JONSWAP (Joint North Sea Wave Project) spectrum. This is given by

$$S(\omega) = 155\frac{H_s^2}{T_1^4\omega^5}e^{-944/(T_1^4\omega^4)}(3.3)^Y(\mathrm{m^2s}) \tag{5}$$

where

$$Y = e^{-[(0.191\omega T_1-1)/(\sqrt{2}\sigma)]^2}$$

and

$$\begin{aligned}\sigma &= 0.07 \quad \text{for} \quad \omega \leq 5.24/T_1 \\ &= 0.09 \quad \text{for} \quad \omega > 5.24/T_1\end{aligned}$$

Another spectrum is the modified Pierson-Moskowitz spectrum which is given by

$$S(\omega) = H_s^2 T_1 \frac{0.11}{2\pi}\left(\frac{\omega T_1}{2\pi}\right)^{-5} e^{-0.44(\omega T_1/2\pi)^{-4}} \tag{6}$$

Spectrum of directional waves are obtained by

$$S(\omega,\theta) = S(\omega)D(\theta). \tag{7}$$

An example of $D(\theta)$ is

$$\begin{aligned}D(\theta) &= \frac{2}{\pi}\cos^2\theta, \quad -\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi \\ &= 0 \quad \text{otherwise}\end{aligned} \tag{8}$$

1.1 Field equation

Ocean surface waves induce a velocity field that below the surface is governed by the Laplace equation. The velocity field is obtained by the gradient of a potential denoted by ϕ. In Fourier space the Laplace equation becomes

$$\hat{\phi}_{yy} - |\mathbf{k}|^2\hat{\phi} = 0 \tag{9}$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ, giving solutions of the form

$$\hat{\phi} = C_0 e^{\pm|\mathbf{k}|y} \tag{10}$$

where C_0 is a complex constant. In deep water this gives $\hat{\phi} = C_0 e^{|\mathbf{k}|y}$. In water of finite water depth with a bottom at $y = -h$ this gives $\hat{\phi} = C_0 \cosh|\mathbf{k}|(y+h)$. By Fourier inversion we obtain

$$\phi = \int\int C_0 \frac{\cosh|\mathbf{k}|(y+h)}{\cosh|\mathbf{k}|h} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} \tag{11}$$

1.2 Boundary conditions

We consider the linearized boundary conditions. The free surface position is determined by $y = \eta$ or, alternatively, $y - \eta = 0$. The kinematic boundary condition expresses that the fluid velocity normal to the surface equals the normal velocity of the surface boundary which means that $D(y-\eta)/Dt = 0$, where $D/Dt = \partial/\partial t + \mathbf{v} \cdot \nabla$, giving, for linear waves

$$\eta_t = \phi_y \quad \text{at} \quad y = 0. \tag{12}$$

The pressure p of the fluid is given by the Bernoulli equation, i.e. $p = -\rho(\phi_t + \frac{1}{2}|\nabla\phi| + gy)$. The dynamic boundary condition expresses that $Dp/Dt = 0$, giving, for linear waves

$$\phi_t + g\eta = 0 \quad \text{at} \quad y = 0. \tag{13}$$

Combination of (12) and (13) gives $\phi_{tt} + g\phi_y = 0$ at $y = 0$. Assuming periodic motion of frequency ω the potential reads $\phi = \operatorname{Re} D_0 e^{\mathrm{i}\omega t}$, giving

$$-\omega^2\phi + g\phi_y = 0 \tag{14}$$

at $y = 0$. Combination of (11) and (14) gives the dispersion relation for ocean surface gravity waves

$$\omega^2 = gk \tanh kh. \tag{15}$$

where $k = |\mathbf{k}|$.

The wave's propagation speed, commonly termed the wave speed, is defined by $c = \omega/k$.

1.3 Example

In infinite water depth the dispersion relation becomes $\omega^2 = gk$. The wave speed becomes $c = g/\omega = \sqrt{g/k}$.

1.4 Example

In shallow water $\tanh kh \simeq kh$. The dispersion relation becomes $\omega^2 = gk^2h$. The wave speed becomes $c \simeq \sqrt{gh}$.

1.5 Example

In moderately shallow water $\tanh kh \simeq kh - \frac{1}{3}k^3h^3$. With this approximation the wave speed becomes $c \simeq \sqrt{gh}(1 - \frac{1}{6}k^2h^2)$.

1.6 Example

The period $T = 2\pi/\omega$ of ocean surface waves are connected to the wavelength $\lambda = 2\pi/k$ through the dispersion relation (15). In water of infinite depth we obtain $\lambda = gT^2/2\pi$, which can be used to evaluate that waves of period 10 sec. have a wavelength of (about) 160 m, waves of period 5 sec. have $\lambda \simeq 40$ m and waves of period 15 sec have $\lambda \simeq 350$ m.

1.7 Nonlinear Stokes waves in deep water

In the case when the water depth is large, a periodic train of Stokes waves has a wave potential

$$\phi = -\frac{gA}{\omega} e^{ky} \sin(-kx + \omega t) + O(A^4 k^4), \tag{16}$$

and the nonlinear dispersion relation reads

$$\omega^2 = gk(1 + A^2 k^2) + O(A^4 k^4). \tag{17}$$

The leading terms of the surface elevation reads

$$\eta = A \cos\chi + \tfrac{1}{2} A^2 k \cos 2\chi + \tfrac{3}{8} A^3 k^2 \cos 3\chi + ..., \quad \chi = -kx + \omega t \tag{18}$$

For ocean waves in deep water the approximations above are excellent. This is e.g. why variants of the nonlinear Schrödinger equation are useful for prediction of evolution of ocean waves within the time window of the evolution of the Benjamin-Feir instability.

Wave induced velocities are given by

$$(u, v) = (\phi_x, \phi_y) = \frac{gAk}{\omega} e^{ky} (\cos\chi, -\sin\chi), \quad \chi = -kx + \omega t \tag{19}$$

1.8 Example

Waves of period 10 sec has a wavenumber of (about) 0.4 m^{-1}. If the height is 10 m, the wave slope is $Ak = 0.2$ which is a very strong wave train on the ocean.

According to (17) the wave length is increased by 4 per cent. The same increase is true for the wave speed.

1.9 Example

The Draupner wave, which is a single large event, the strongest that has been documented at sea, has an estimated wave slope of 0.39.

1.10 Linear shallow water waves

The linear shallow water equations read

$$\eta_{tt} - c_0^2 \eta_{xx} = 0 \tag{20}$$

where $c_0 = \sqrt{gh}$.

1.11 Weakly nonlinear shallow water waves

Shallow water waves in one-directional propagation in a fluid layer of depth h is modelled by the Korteweg-de Vries equation, which reads

$$\eta_t + \sqrt{gh}\, \eta_x + \tfrac{3}{2} \sqrt{\tfrac{g}{h}}\, \eta\eta_x + \tfrac{1}{6} h^2 \sqrt{gh}\, \eta_{xxx} = 0 \tag{21}$$

1.12 Soliton solution

The KdV equation (21) has solitary wave solutions of the form

$$\eta(x,t) = H\,\mathrm{sech}^2\,k(x-ct) \tag{22}$$

where

$$k = \frac{1}{h}\sqrt{\frac{3H}{4h}} \tag{23}$$

$$c = \sqrt{gh}\Big(1+\tfrac{1}{2}\tfrac{H}{h}\Big) \tag{24}$$

1.13 Cnoidal waves

Periodic nonlinear waves in shallow water may be approximated by the theory for cnoidal waves. For the elevation, the first term in a series expansion is obtained by Fenton (1990) by

$$\frac{\eta}{h_{trough-depth}} = 1 + \epsilon\,\mathrm{cn}^2(u|m) + ... \tag{25}$$

where $u = \alpha(x - ct/h_{trough-depth})$, $\alpha = \sqrt{3\epsilon/4m} + ...$, $\epsilon = H/h_{trough-depth}$, H wave height.

$\mathrm{cn}(u|m)$ denotes one of the Jacobian elliptical functions defined by

$$\mathrm{cn}(u|m) = \cos\phi, \quad u = \int_0^{\phi} \frac{d\theta}{(1-m\sin^2\theta)^{\frac{1}{2}}} \tag{26}$$

where $m \leq 1$. For $m = 0$ $\mathrm{cn}(u|0) = \cos u$. For $m = 1$ $\mathrm{cn}(u|1) = \mathrm{sech} u$.

Fenton (1990) has studied the ranges of application of Stokes theory and cnoidal theory, for waves in shallow water.

1.14 Ursell number

The Ursell number evaluates the ratio between the nonlinearity and dispersion parameters and is given by

$$U_r = \frac{H}{h}\Big(\frac{\lambda}{h}\Big)^2 \tag{27}$$

where H denotes the wave height, h water depth and λ wave length.

1.15 Example

Long waves in water of depth 25-45 m have period 12 s and wave height 10 m. The wave frequency becomes $\omega = 0.357\ \mathrm{s}^{-1}$ and $\omega^2 h/g = 0.325$ with $h = 25$ m, which gives $kh = 0.603$. The relative wave length becomes $\lambda/h \simeq 10$. The Ursell number becomes 40 which tells that Stokes wave theory (5th order) is still applicable. (Illustration).

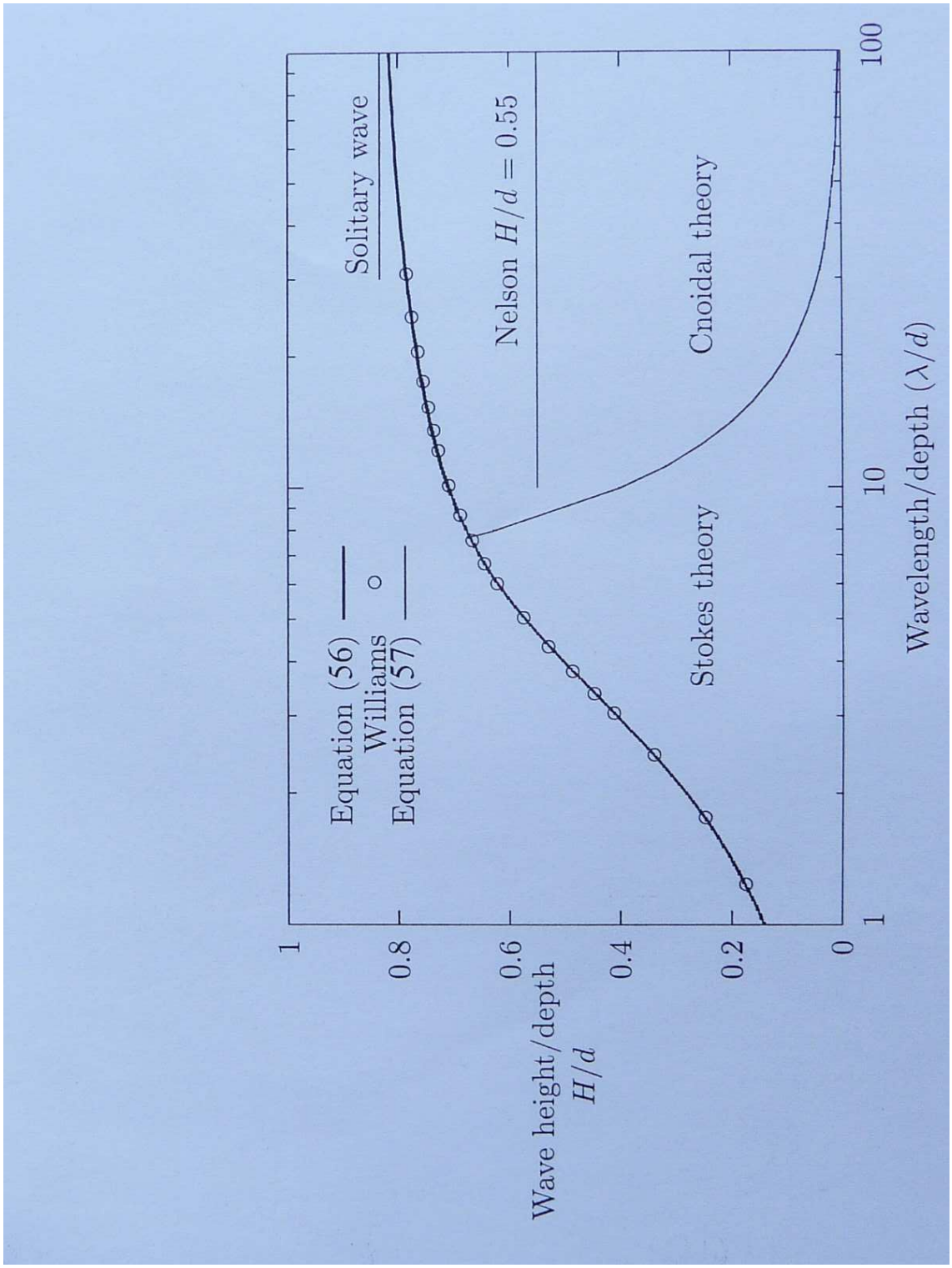

Figure 1: Application ranges. Stokes (5th-order) and cnoidal wave theories. From Fenton, 1990.