An incompressible SIMPLE method and its application on discontinuous grids

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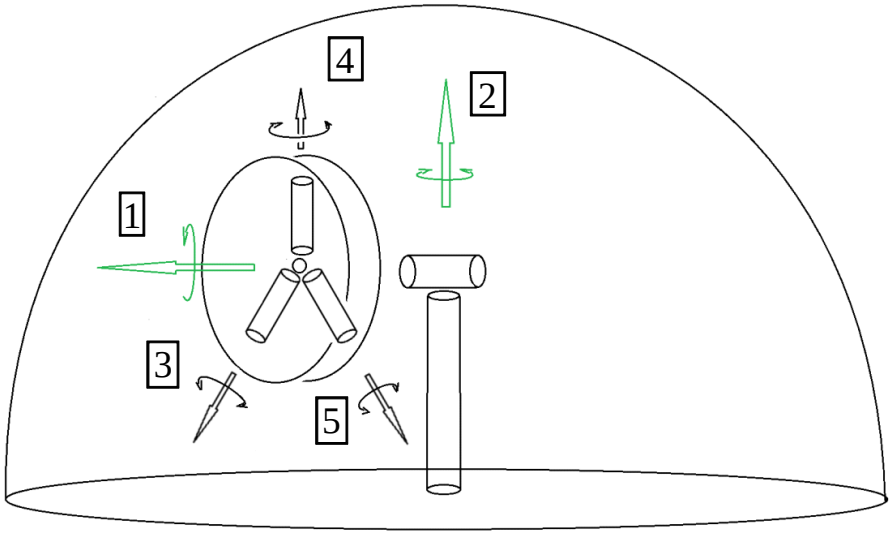
June 4, 2013

Overview

Introduction

- Most of computational codes for solution of incompressible Navier-Stokes equations today are based on the SIMPLE method and its modifications.
- One of the applications of the method is computation of flow around moving objects such as rotating machineries.



Wind turbine as viewed by engineers ...

- For these kinds of problems block-structured grids with discontinuous interfaces may be employed.

Introduction

Original method developed in EllipSys2D/3D code is based on:

- Collocated grids: flow field variables are stored in cell centres.
- Continuity equation is approximated using interpolation of momentum equations on cell faces.
- Pressure gradient at cell face is computed based on the Rhie-Chow interpolation.
- Time terms can be interpolated using the Revised Rhie-Chow interpolation.
- Pressure-correction equation is solved using Multigrid method.

The purpose of the current work is to develop the computational method in the EllipSys2D/3D code for applications on grids with nonconformal interfaces.

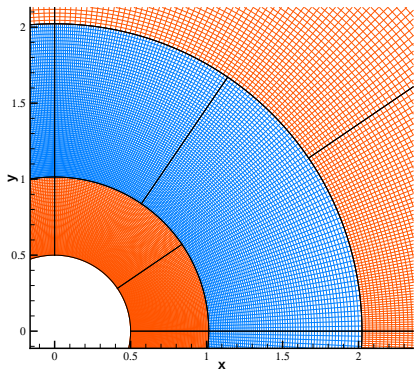
The main requirements are:

- Efficiency
- Accuracy
- Consistency with the original solver

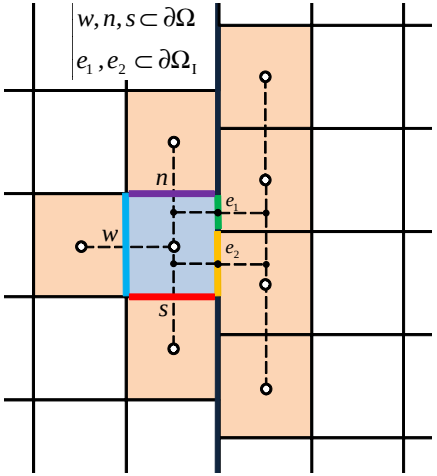
Discretisation on non-conformal grids: Introduction and Requirements

At discontinuous interface:

- The grid mismatch may be arbitrary in real life computations.
- The most reliable approach for the problem is conservative finite-volume scheme.
- Any control volume flux assigned to a CV may be expressed as sum of CV fluxes through the segments which the face consists of.
- Discretisation at interfaces makes computational matrixes irregular.
- For solution of Momentum equations using Gauss-Seidel or Red-Black the solvers have to treat the interface terms preserving implicit nature of the algorithm.
- Mass fluxes have to be separately computed at each interface segment based on Rhie-Chow interpolation.
- Pressure-Correction equation has the matrix with irregular structure: multigrid solver has to be able to solve such system of equations efficiently and accurately.



(a) Grid with two discontinuous interfaces



(b) Cells at interface

Discretisation on interface: Finite-Volume approximation

The idea is:
To preserve unique contribution of finite volume fluxes into control volume integrals at interface.

The discretisation of Navier-Stokes equations is based on:

- Conservative second order finite volume method
- Flow variables defined at each cell face segments
- Auxiliary nodes introduced with corresponding polinomial interpolation

The finite volume fluxes at interface are composed of convective, diffusion and the pressure source terms:

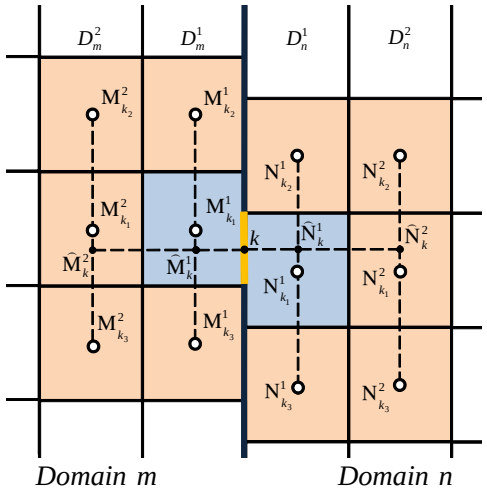


Figure 1 : Auxiliary nodes used for finite-volume approximation.

$$\int_{\Omega} \frac{\partial \rho u}{\partial t} \mathrm{d}V + \int_{\partial\Omega'} \rho \vec{u}\vec{\hat{n}} u \mathrm{d}S = \int_{\partial\Omega'} \mu \vec{\nabla} u \vec{\hat{n}} \mathrm{d}S - \int_{\partial\Omega'} \vec{\nabla} p \vec{i}_x \mathrm{d}V$$

Each of the terms has different contribution into computational matrix and the source term of Momentum equations.

Discretization on interfaces: System of momentum equations

The resulted system of momentum equations is:

$$\widetilde{A_P} u_P^{*,m+1} + \sum_{k \in \partial\Omega \cup \partial\Omega_{\mathrm{I}}} A_k^m u_k^{*,m+1} = S_P^u + \frac{\rho \mathrm{d}V}{\mathrm{d}t}(2u_P^n - 0.5u_P^{n-1}) - \sum_{k \in \partial\Omega \cup \partial\Omega_{\mathrm{I}}} p_k^{*,m+1} \vec{i}_x \cdot \vec{\hat{n}} \mathrm{d}S_k$$

$$\widetilde{A_P} v_P^{*,m+1} + \sum_{k \in \partial\Omega \cup \partial\Omega_{\mathrm{I}}} A_k^m v_k^{*,m+1} = S_P^v + \frac{\rho \mathrm{d}V}{\mathrm{d}t}(2v_P^n - 0.5v_P^{n-1}) - \sum_{k \in \partial\Omega \cup \partial\Omega_{\mathrm{I}}} p_k^{*,m+1} \vec{i}_y \cdot \vec{\hat{n}} \mathrm{d}S_k$$

To solve the system of momentum equations with irregular computational matrices the Block Jacoby and Gauss-Seidel solvers were modified based on:

- Computation of interface matrix terms
- Contribution of the matrix terms into residual update

The approach is based on:

- Cumbersome expression of convective and diffusion terms at interface (derived, not presented)

But as the result:

- The additional computational effort for solution of momentum equations is nevertheless small enough

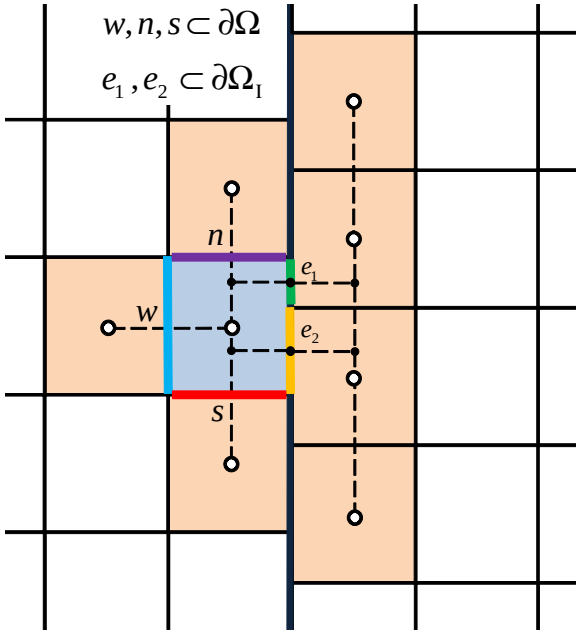


Figure 2 : Control volume

Preprocessor and Parallelisation: Parallelisation

Parallelisation is composed of speccially designed preprocessor and new interface communicators:

- Prior to computations:
  - Interfaces identification based on block groups
  - Grouping of processors on grounds of their connection to the same interface
  - Assignment of unique MPI communicator for each processor group
- During computations:
  - Data broadcasting within each processor group using the assigned MPI communicator

As the result:

- Preprocessor prepares data for more then one interface in general
- Communications are carried out within groups of processors assigned to each interface
- Same process may handle several interfaces
- The computational overhead is small

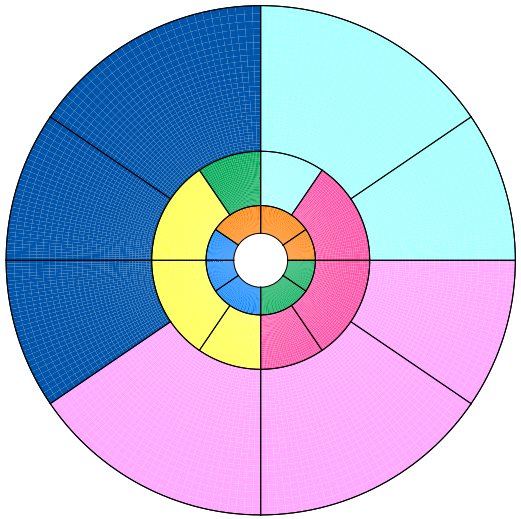
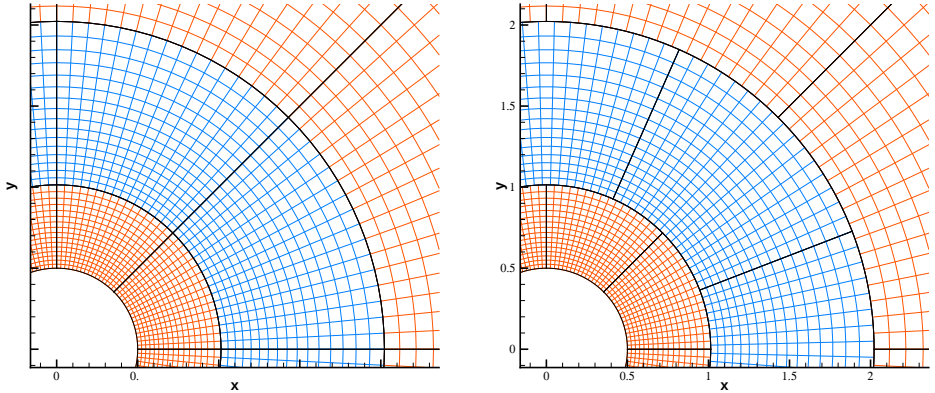


Figure 3 : Processor groups.

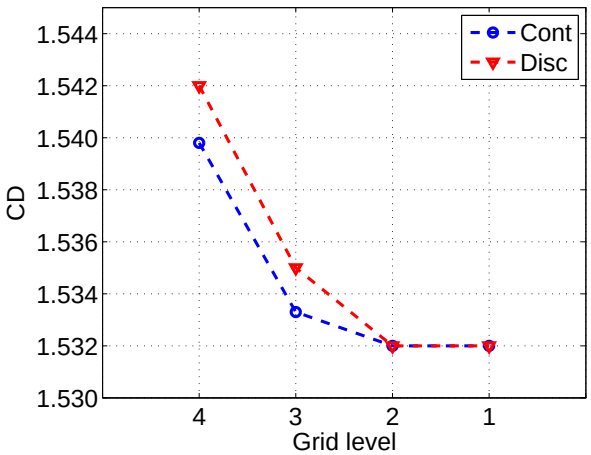
General performance: Steady flow around cylinder



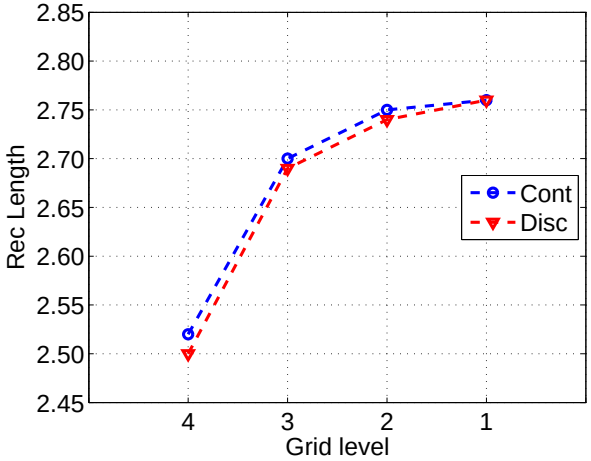
(a) Continuous grid. (b) Discontinuous grid.

Figure 4 : Cont. and disc. grids with equidistant cells along interface with 128x64 cells.

Drag coefficient and Recirculation zone length for grids 1) 512x256 cells, 2)256x128 cells, 3) 128x64 cells and 4) 64x32 cells

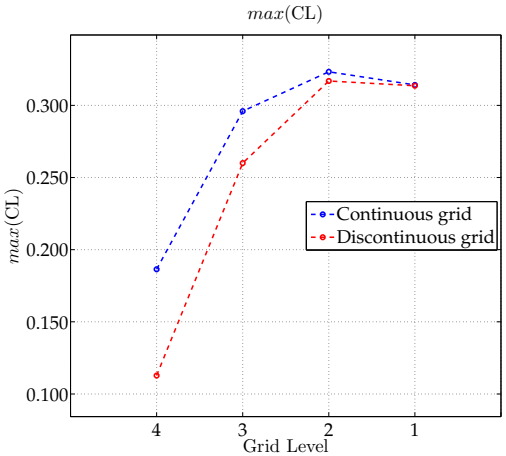


(a) Drag coefficient

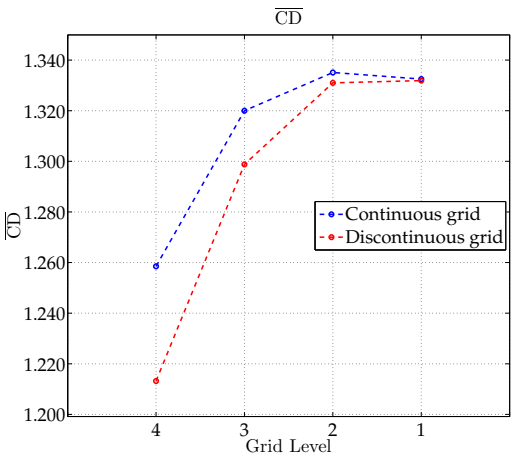


(b) Recirculation length

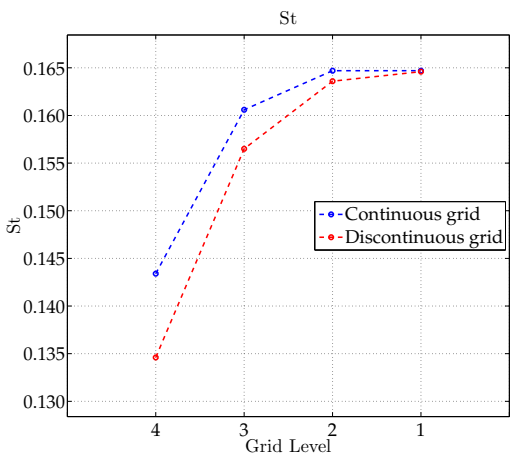
General performance: Unsteady flow around cylinder (1/2)



(c) Max lift coeffiecient, $max(\mathrm{Cl})$



(d) Drag coeffiecient, Cd



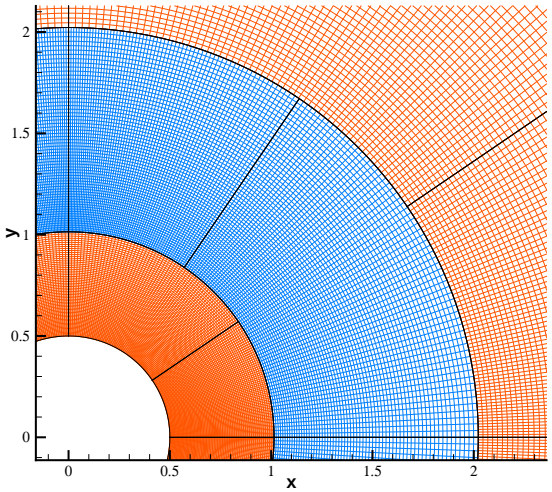
(e) Strouhal number, St

Figure 5 : Solution on cont. and disc. grids with 2 cells/cell grid mismatch and $dt = 2 \cdot 10^{-2}$.

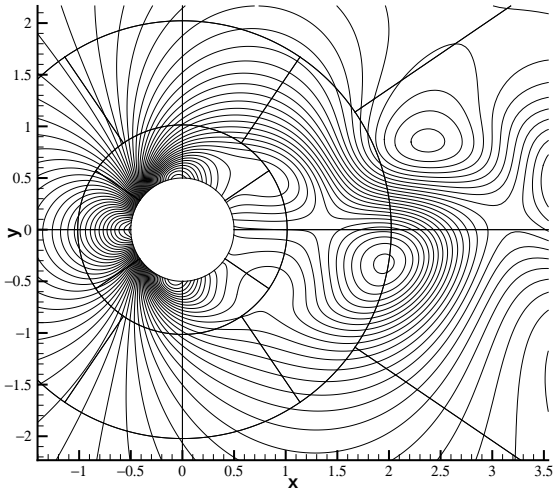
Grid	St	$\overline{\mathrm{CD}}$	$max(\mathrm{CD})$	$min(\mathrm{CD})$	$max(\mathrm{CL})$	$min(\mathrm{CL})$
64x32	0.1346	1.2132	1.2146	1.2119	0.1128	-0.1128
128x64	0.1565	1.2988	1.3056	1.2919	0.2600	-0.2599
256x128	0.1636	1.3310	1.3400	1.3219	0.3169	-0.3169
512x256	0.1646	1.3319	1.3405	1.3233	0.3136	-0.3136

Table 1 : Solution dependence on grid resolution on disc. grids with 2 cells per cell.

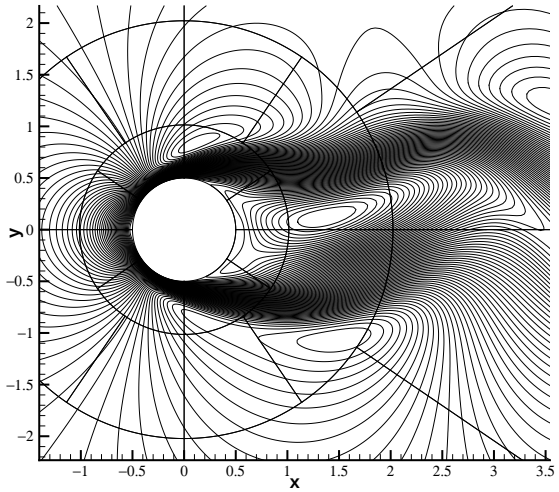
General performance: Unsteady flow around cylinder (2/2)



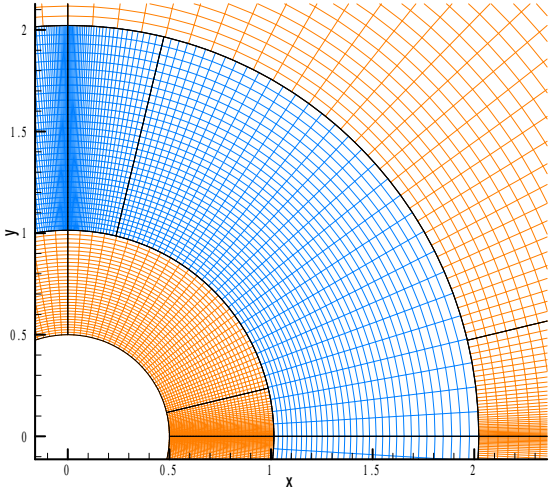
(a) Grid mismatch: 4 cells per cell



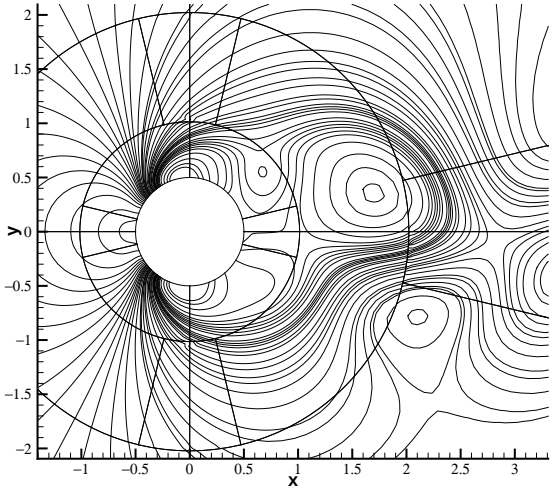
(b) Pressure contours



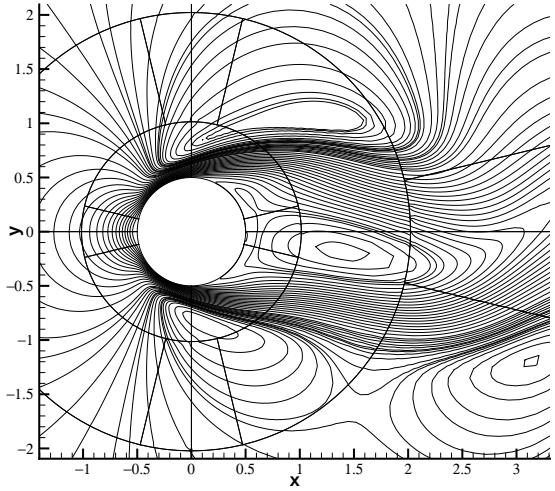
(c) u velocity component contours



(d) Grid mismatch: 16 cells per cell



(e) Pressure contours



(f) u velocity component contours

Discretisation: Pressure-Correction equation

Solution of pressure-correction equation (PC) is one of the main issues of SIMPLE-like methods. Following Patankar the pressure-correction equation can be expressed as:

$$\sum_{k\in\partial\Omega\cup\partial\Omega_{\mathrm{I}}} \left(F_k^C + F_k^*\right) = 0 \quad \text{which is} \quad a_p p_p^C + \sum_{k\in\partial\Omega\cup\partial\Omega_{\mathrm{I}}} a_k p_k^C = -\sum_{k\in\partial\Omega\cup\partial\Omega_{\mathrm{I}}} F_k^*$$

The discretisation of PC equation on discontinuous interfaces is important as it determines:

- General efficiency of the Multigrid solver
- Accuracy of continuity equation
- Pressure-velocity coupling and general continuity of flow field

To preserve accurate and efficient solution of the Multigrid solver the discretization of the PC equation is based on:

- Conservative approach: Mass fluxes at interface segment are computed consistently to their equal contributions into the source terms.
- Second order mass flux interpolation F_k^*, consistent with momentum equations.
- Robust correction flux approximation F_k^C with relatively simple matrix composition of Poisson equation without influence on accuracy of converged solution.

Discretisation: Pressure-Correction equation

Mass fluxes are computed by expressing them through the momentum equations:

$$F_k^{*,m+1} = \rho_k \vec{\tilde{\tilde{u}}}_k \mathrm{d}\vec{S}_k - \overline{\left[\frac{\rho \mathrm{d} S_P}{\widetilde{A_P}}\right]}_k \left(p^{*,m+1}_{\widehat{N_k^1}} - p^{*,m+1}_{\widehat{M_k^1}}\right) \mathrm{d}S_k + \overline{\left[\frac{\frac{\rho \mathrm{d} V}{dt}}{\widetilde{A_P}}\right]}_k \left(2F_k^n - 0.5F_k^{n-1}\right)$$

Correction flux F_k^* is expressed using first order approximations: $p^C_{\widehat{M_k^1}} = p^C_{M_k^1}$ and $p^C_{\widehat{N_k^1}} = p^C_{N_k^1}$.

$$F_k^C = -\overline{\left[\frac{\rho \mathrm{d} S_P}{\widetilde{A_P}}\right]}_k \left(p^C_{\widehat{N_k^1}} - p^C_{\widehat{M_k^1}}\right) \mathrm{d}S_k = a_k \left(p^C_{N_k^1} - p^C_{M_k^1}\right)$$

The pressure-correction equation is:

$$a_p p_p^C + \sum_{k \subset \partial\Omega \cup \partial\Omega_\mathrm{I}} a_k p_k^C = \sum_{k \subset \partial\Omega \cup \partial\Omega_\mathrm{I}} F_k^*$$

The mass fluxes as defined above are equally evaluated at two sides of the interface, which enables us to solve the PC equation efficiently using Multigrid method

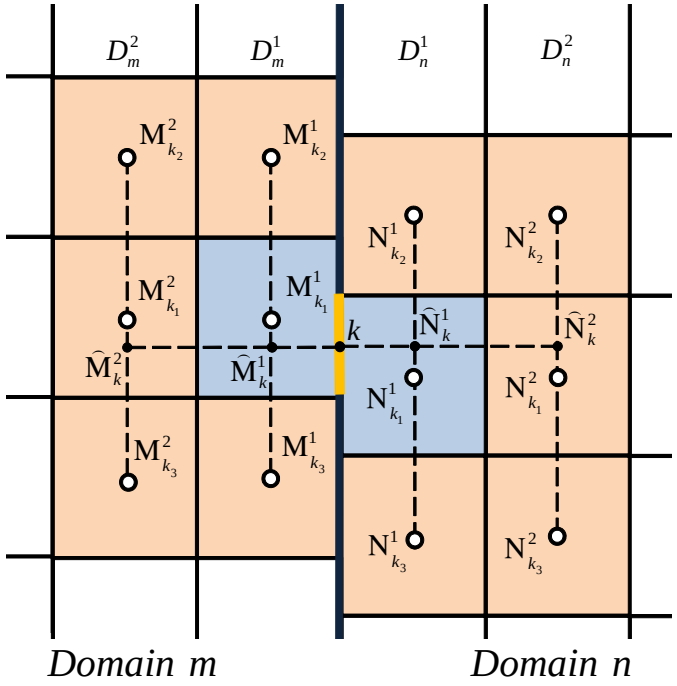


Figure 6 : Mutligrid V-cycle

Multigrid method on discontinuous grid: Outline

Multigrid method composed of the following components has been extended to nonconformal grids:

- Coarse grid solvers based on Krylov-type methods, modified for applications on nonconformal grids:
  - IBLU-preconditioned Conjugate gradient method
  - IBLU-preconditioned Restarted GMRES method developed based on Josef Saad method.
- Incomplete Block-LU factorisation as relaxation with adjusted BC at discontinuous interfaces
- Coarsening grid operators based on mass flux conservation law between grid levels at interface
- Prolongation operator based on bilinear interpolation of finer grids values at interface

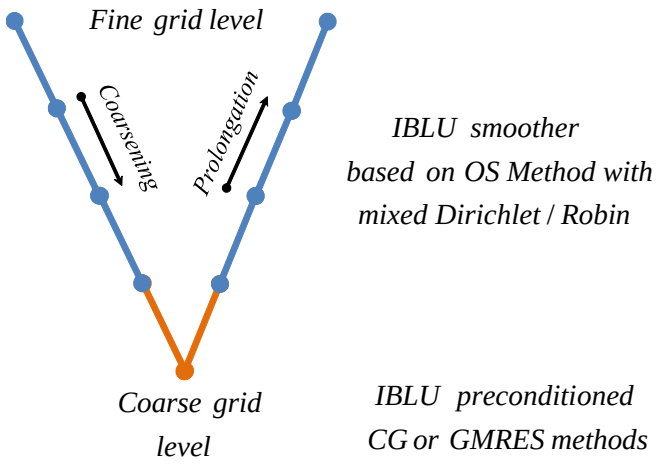


Figure 7 : Mutligrid V-cycle

- Weakly overlapped domain decomposition based on kind of Optimised Schwarz method with:
  - Robin/Dirichlet BC at internal block boundaries
  - Robin parameter depending on grid geometry

Multigrid: Coarsening operator

Correction mass flux is conserved between levels

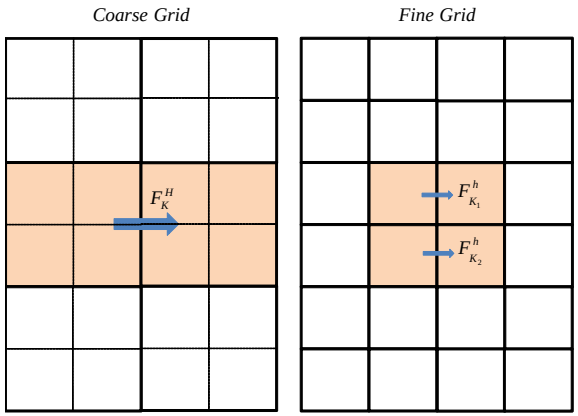


Figure 8 : Continuous grid

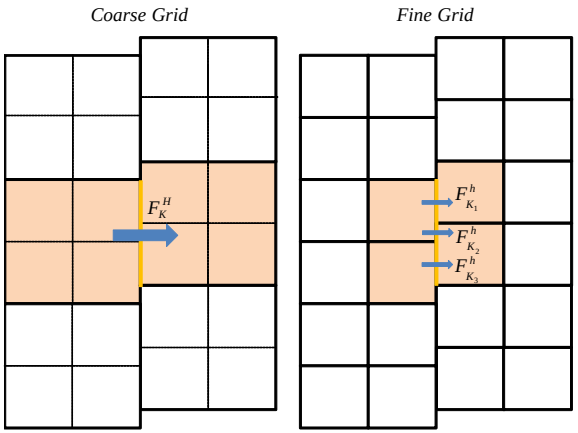
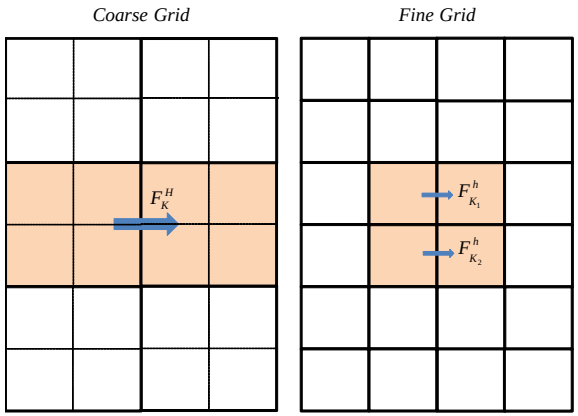


Figure 9 : Disc. grid

$$a_k^H = 1/2(a_{k_1}^h + a_{k_2}^h)$$

Multigrid: Coarsening operator

Correction mass flux is conserved between levels



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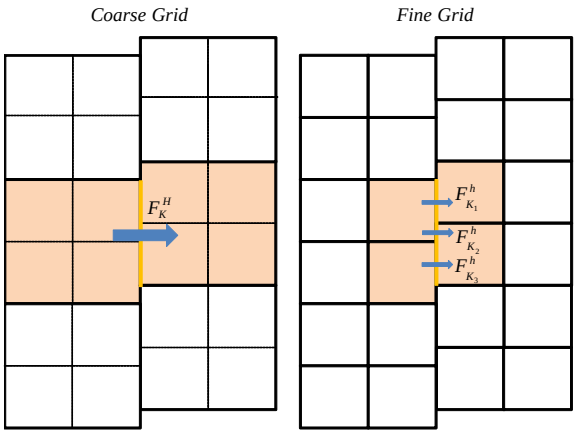

Figure 9 : Disc. grid

$$a_k^H = 1/2(a_{k_1}^h + a_{k_2}^h)$$

$$\begin{aligned} F_k^{C,H} &= \sum_{i \subset \partial\Omega} F_{k_i}^{C,h} \\ a_k^H \Delta p_k^{C,H} &= \sum_{i \subset \partial\Omega} a_{k_i}^h \Delta p_{k_i}^{C,h} \\ a_k^H &= 1/2 \sum_{k_i \subset k} a_{k_i}^h \end{aligned}$$

Multigrid: Prolongation operator

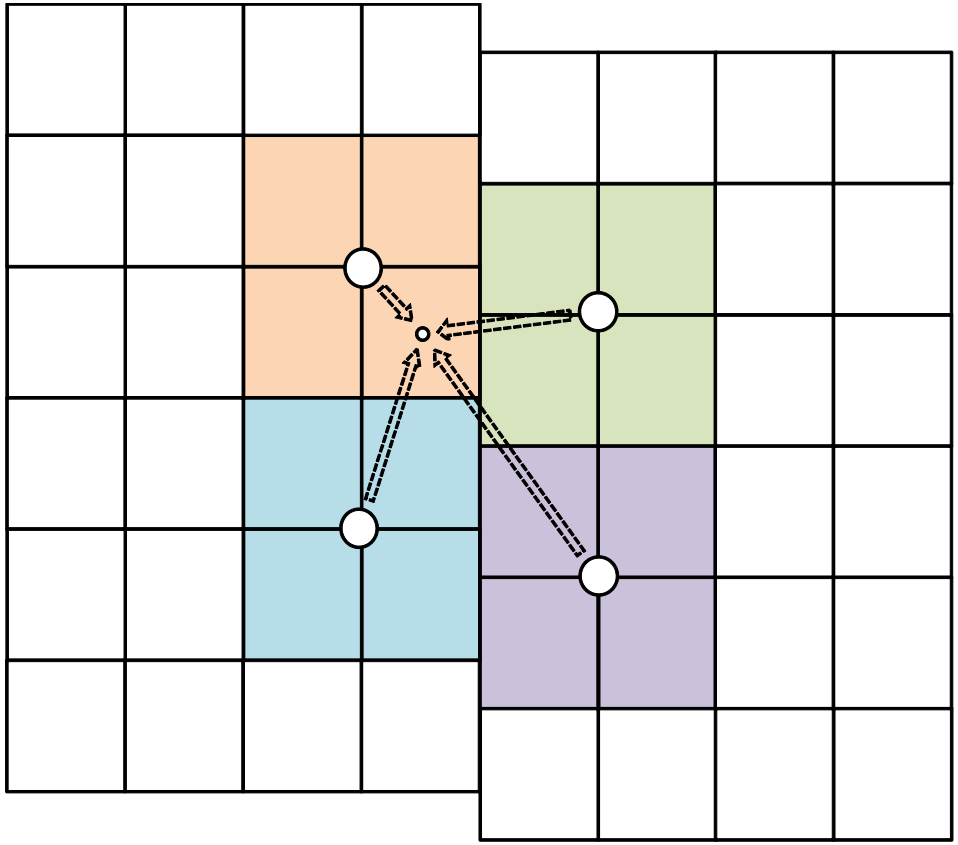

Figure 10 : Prolongation operator

Fine grid values are computed based on:

Bilinear interpolation using coarse grid values at nonconformal interface

Multigrid: (1/2) Relaxation using Optimised Schwarz method

According to theory the Robin BC employed for the IBLU relaxation scheme in the original EllipSys2D/3D solver appeared to have optimal conditions:

- Is strictly positive, Lions 1990
- Decreases for increasing distance to cross point, Gander 2006
- Optimally scales at cross points as $O(h^{-1})$, Gander 2012

Practically:

- The OS method employed in IBLU relaxation scheme of the Multigrid method in EllipSys code on continuous grids has proved to be a very robust method for last fifteen years.

Multigrid: (2/2) Relaxation using Optimised Schwarz method

To extend the IBLU-factorisation scheme to nonconformal grids it has been modified by:

- computing the appropriate BC at each interface segment

The boundary condition contribute into diagonal term as:

$$a_p = -\sum_{f \in \partial\Omega} a_f - \sum_{k \in \partial\Omega_I} a_k B_k(\widehat{\lambda_k})$$

- Evaluation of parameter B_k at each interface side at each interface segment k is:

$$B_k^{D_m}(\lambda_k) = \max(0.5, 1 - \frac{d1^{D_m}}{dd^{D_m}} - \frac{d1^{D_m}}{d2^{D_m}})$$

- Averaging of the B_k neighbour values:

$$B_k = \frac{1}{2}\left(B_k^{D_m} + B_k^{D_n}\right)$$

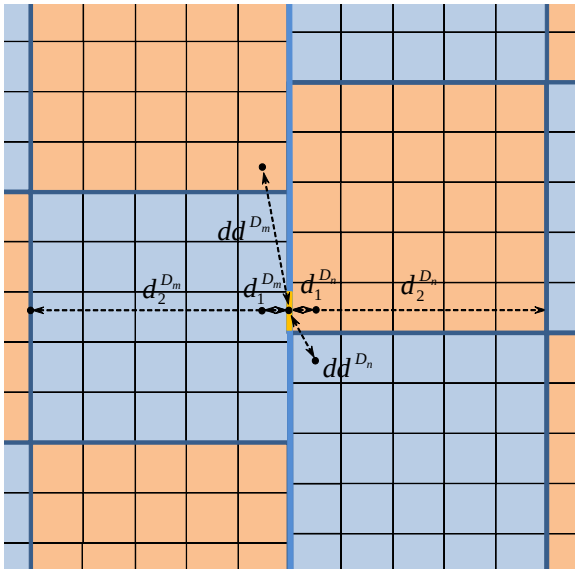

Figure 11 : Robin parameter at interface

The averaging coefficient may be different from 0.5 to preserve $O(h^{-1})$ scaling of λ_k in areas close to cross points.

Multigrid: Convergence analysis

Convergence rate comparisons based on CS and OS domain decompositions for:

- 1) IBLU factorization as solver;
- 2) Multigrid method with IBLU factorization as smoother and with IBLU-preconditioned Conjugate Gradient method as Coarse Grid correction.

Grid type	IBLU as solver		Multigrid method			
	CS	OS	CS	OS	CS	OS
Continuous	17619	6036	328	229	1662	1068
Discontinuous	17451	5991	331	231	1677	1077
	Iterations		Multigrid cycles		IBLU sweeps	

Table 2 : Convergence rate of solvers based on Classical Schwarz and Optimized Schwarz domain decompoistion methods

As the result the convergence speeds on continuous and discontintuous grids are identical:

- The extension of the Multigrid method to nonconformal grids is proved to be correct and efficient.

Conclusions on general computational method

Computational method has been developed for application on discontinuous grids based on:

- Conservative Finite Volume approach
- Conservative Mass Flux interpolation
- Pressure-Velocity Coupling
- Well posed Multigrid Method
- Parallelised interface data exchange

The performance shows that the method is:

- Accurate
- Robust
- Fast

Current work

1) In the scope of the moving grids:
A new alternative to the Revised Rhie-Chow interpolation has been developed which:

- (+) does not need to employ the former mass fluxes for evaluation of time terms in expression of mass fluxes.
- (+) has the solution weakly dependent on time step.
- (+) has no pressure wiggles at decreasing time step.
- (+) may be employed for moving/sliding/overset grids.
- (-) has a lower convergence rate in comparison with the Revised Rhie-Chow interpolation

2) On modification of the scheme with Revised Rhie-Chow interpolation independent on time step and relaxation parameter at convergence:

- The original scheme in EllipSys2D/3D solver posses relatively high time step dependence on coarser grids.
- The new scheme was derived (based on ideas of Pascau, 2012) independent on relaxation parameter and time step at convergence

Thank you for attention!